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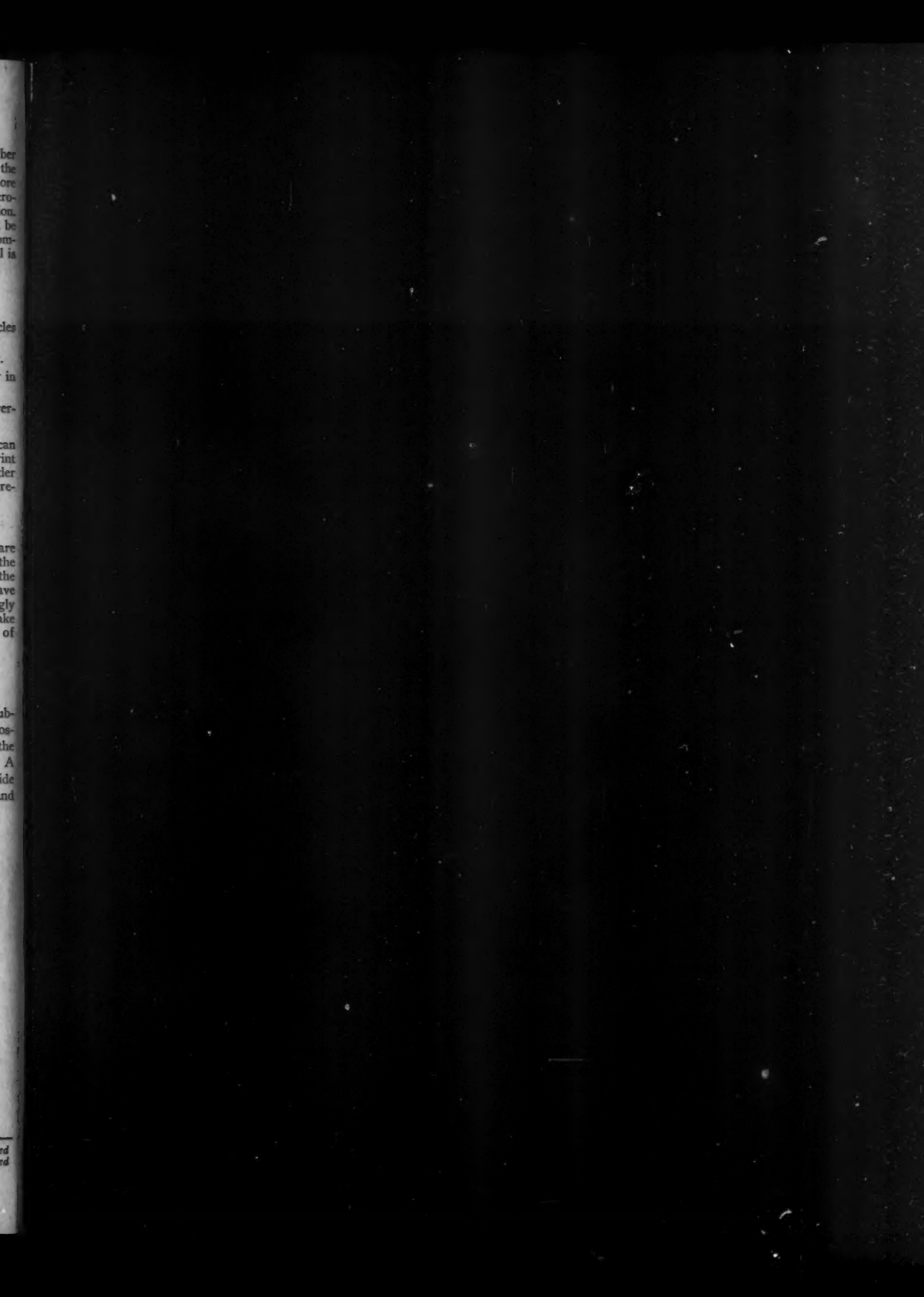

Mathematical Reviews

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ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Carruth, Philip W. Arithmetic of ordinals with applications to the theory of ordered Abelian groups. Bull. Amer. Math. Soc. **48**, 262–271 (1942). [MF 6404]

Hessenberg and Hausdorff [see Hausdorff, Mengenlehre, 3rd ed., de Gruyter, Berlin-Leipzig, 1935, pp. 68, 70] have defined certain "natural" sums and products of ordinal numbers. The author gives new information concerning these sums and products. The sum is shown to be the smallest natural sum and the best bound for the order type of the join of two well ordered subsets of an ordered set. The product is shown to be the smallest natural product and the best bound for the order type of a certain rectangular array. There is an application to the theory of ordered Abelian groups. *D. Montgomery* (Northampton, Mass.).

Faedo, Sandro. Il principio di Zermelo per gli spazi astratti. Ann. Scuola Norm. Super. Pisa (2) **9**, 263–276 (1940). [MF 5466]

The aim of this paper is to avoid the general principle of selection for collections of closed sets by finding a certain rule for determining a point in every closed set. For Euclidean spaces this is trivial. The author accomplishes it for a rather general type of (not necessarily compact) spaces, after having done it previously [Atti II° Congr. Un. Mat. Ital., Bologna, 1940] for the Hilbert space. In the type of spaces now considered the points are also sequences of real numbers; the distance depends continuously on the coordinates and satisfies a generalized triangle inequality.

A. Rosenthal (Albuquerque, N. M.).

Kurepa, Georges. Transformations monotones des ensembles partiellement ordonnés. Revista Ci., Lima **42**, 827–846 (1940); **43**, 483–500 (1941). [MF 4347]

The author considers a special kind of partially ordered system, a ramified table, that is, a system in which the predecessors of any fixed element form a well-ordered system. In his thesis [Publ. Math. Univ. Belgrade **4**, 1–138 (1935)] the author studied these systems in his attack on Souslin's problem. In the present paper he continues their study using the property of existence or nonexistence of real-valued increasing functions. Certain typographical difficulties hinder the reading of the first part of the paper. In the second part the author also obtains certain results about ramified tables whose elements are: (a) closed subsets of a topological space with a countable base, or (b) linear sets of positive measure. *J. W. Tukey.*

Kurepa, G. Une propriété des familles d'ensembles bien ordonnés linéaires. Studia Math. **9**, 23–42 (1940). (French. Ukrainian summary) [MF 5254]

A partially ordered system is normal if either (a) it is finite or (b) it contains a subsystem F of the same power with the property that for each $a \in F$ the subsystem of elements of F comparable with a is fully ordered. If E is any fully ordered system, then E is the system of its well ordered subsystems ordered by the relation "is a proper initial segment of." E is a normal ramified table [cf. the preceding review], but it is not clear that this is true for all its subsystems. In this paper the author attacks four special cases of this problem: (I) E is countable, hence similar to a set of rational numbers; (II) every interval of E is the union of part of a fixed countable family of intervals of E, hence E is similar to a subset of $(0, 1)$; (III) E possesses a countable dense subsystem, hence is similar to a subset of the lexicographical product of $(0, 1)$ by a fully ordered system with two elements; (IV) E verifies Souslin's condition, that is, every family of distinct intervals of E is countable.

The author succeeds in proving the normality of every subsystem of E in the first three cases. The fourth case remains open, but the following equivalence is proved: If E is an ordered system, each of whose families of disjoint intervals is countable, then E has a countable dense subsystem if and only if every uncountable family of well ordered subsystems of E contains an uncountable subfamily of systems no one of which is a proper initial segment of another. *J. W. Tukey* (Princeton, N. J.).

Jankoff, W. Sur l'uniformisation des ensembles A. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 597–598 (1941). [MF 4468]

Lusin [Mathematica **4**, 54 (1930)] has demonstrated that every plane analytic set can be uniformized by a set E (that is, E contains precisely one point on each vertical line which intersects E) which belongs to a field generated by sets A and CA. This note demonstrates this result by another method. In fact, it is shown that the uniformizing set may be an $A_{\sigma\delta}$, where A_σ designates the difference of two analytic sets, and that the function defining the uniformizing set may be measurable. *D. Montgomery.*

Koslova, Z. Sur les ensembles plans analytiques ou mesurables B. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 479–500 (1940). (Russian. French summary) [MF 4086]

The author proves results of the following types: If E is a plane analytic set cut by each vertical line in a set with property A, (1) then E can be enclosed in a Borel set with the same property; (2) then E is the sum of a sequence with property A of disjoint uniform Borel sets. Results (1) and (2) are proved where A is "well-ordered (reducible) of type not greater than α." Result (1) alone is proved where A is "clairsemé of type not greater than α" and "perfect and nondense." *J. W. Tukey.*

Arsenin, V. Sur la nature des projections de certains ensembles mesurables B. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 403–410 (1940). (Russian. French summary) [MF 4081]

The author proves the following theorem: If E is a Borel measurable set in the plane and H the set of ordinates for

which vertical lines cut E in a nonempty F_σ, then H is the complement of an analytic set. In particular, if each vertical cuts E in an F_σ, then the projection of E is a Borel set. This result includes results of Lusin, Novikoff, Braun and Kunugui. *J. W. Tukey* (Princeton, N. J.).

Keldych, Ludmila. Sur la structure des ensembles mesurables B. C. R. (Doklady) Acad. Sci. URSS (N.S.) **31**, 651–653 (1941). [MF 5238]

The author continues her study [C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 523–525 (1940); these Rev. **2**, 256] of the structure of sets of Borel class α, for any $\alpha<\omega_1$. She proves that every set precisely of class α is a disjoint sum of a "canonical element of class α" and a countable number of canonical elements of lower classes. The original definition of a canonical element of class α involved an A-system defining it [cf. reference above]. She now proves that an element of class α is canonical if and only if it is a universal element of class α on every portion of its closure. An application of an earlier theorem allows one to conclude that any two canonical elements of class α are homeomorphic. *J. W. Tukey* (Princeton, N. J.).

Randolph, John F. Some properties of sets of the Cantor type. J. London Math. Soc. **16**, 38–42 (1941). [MF 5902]

Let $\{a_n\}$ be a sequence with $a_n>0$, $\sum a_n=1$, ϵ_n a sequence with $\epsilon_n=0$ or 1. For a fixed $\{a_n\}$ let S be the set of all sums of the form $\sum\epsilon_n a_n$. A question raised by Kac and answered by the author is: What is a necessary and sufficient condition that the set of distances between the points of S fills the unit interval? It is that $a_n\leqq 2r_n$, where $r_n=\sum_{n+1}^{\infty}a_k$.

A Cantor set C_λ of zero measure is defined on the unit interval by removing from its center the interior of an interval of length λ, then from the centers of each of the two remaining intervals the interior of an interval of length $(1-\lambda)\lambda/2$, from the centers of each of the four remaining intervals the interior of an interval of length $(1-\lambda)^2\lambda/2$, etc.; T_λ is the two-dimensional set (x, y) with x and y both in C_λ. The following results are obtained: For the Carathéodory linear measure LT_λ, if $\lambda\neq 1/2$ then LT_λ is zero or infinity. If $\lambda<1/2$, $LT_\lambda=\infty$. If $0<\lambda\leqq 1/2$, the projection of T_λ on the line $y=x/2$ is an interval of length $3/5^{1/2}$. If $\lambda=1/2$ then $3/5^{1/2}\leqq LT_\lambda\leqq 2^{1/2}$. If $1/2<\lambda<1$, then $LT_\lambda=0$. The Gillespie linear measure GT_λ is ∞ for $0<\lambda<1/2$, positive and finite for $\lambda=1/2$ and zero for $1/2<\lambda<1$. Also $GT_{1/2}\neq LT_{1/2}$, and $3/5^{1/2}\leqq LT_{1/2}\leqq 2^{1/2}<3(2/5)^{1/2}\leqq GT_{1/2}\leqq 2$. *R. L. Jeffery* (Wolfville, N. S.).

Besicovitch, A. S. Relations between concentrated sets and sets possessing property C. Proc. Cambridge Philos. Soc. **38**, 20–23 (1942). [MF 6030]

A linear set E is said to have property C if, to any sequence of positive numbers l_n, there corresponds a sequence of intervals I_n such that $E\subset\sum I_n$ and $|I_n|\leqq l_n$. A nonenumerable linear set E is said to be concentrated about an enumerable set X if any open set containing X contains all of E except for an enumerable subset of E. A concentrated set clearly has property C. This paper settles a question raised by Sierpiński [Fund. Math. **30**, 56–58 (1938)] as to the equivalence of these two classes of sets by exhibiting a set with property C which is not concentrated about any enumerable set. *L. W. Cohen.*

Besicovitch, A. S. A theorem on s-dimensional measure of sets of points. Proc. Cambridge Philos. Soc. **38**, 24–27 (1942). [MF 6031]

A set E is said to be an s-set if its Hausdorff t-dimensional measure is equal to 0 for any $t>s$ and to ∞ for any $t<s$. By methods which are effective in spaces of any dimension and for more general sets, the author proves that any plane Borel s-set can be decomposed into $2^{\aleph_0}$ s-sets. This yields a simple proof of a theorem of Littlewood: In a conformal representation of a circle on a domain bounded by a Jordan curve, to a set of points of linear measure zero on the boundary of the circle may correspond a set of positive measure on the Jordan curve. *L. W. Cohen.*

Sherman, Seymour. A comparison of linear measures in the plane. Duke Math. J. **9**, 1–9 (1942). [MF 6336]

Carathéodory's definition of linear measure LA of a plane set A is undoubtedly the best known generalization of curve length. Several definitions, including Carathéodory's, of linear measure are independent of the Lebesgue theory of measure, but, on the other hand, Steinhaus based a definition of linear measure directly on the Lebesgue theory of measure and integration. Let A be a plane set and $f_A(\rho, \theta)$ the number (finite or $+\infty$) of times the line $x\cos\theta+y\sin\theta-\rho=0$, $\rho\geqq 0$, $0\leqq\theta\leqq\pi$, intercepts the set A, where the ambiguities arising for $\rho=0$ or $\theta=2\pi$ are taken care of in a natural way. If the function f_A is summable on its strip of definition, then the integral $\int\int f_A(\rho,\theta)d\rho d\theta$ over this strip is by definition the Steinhaus linear measure $St(A)$ of A. This measure is invariant under Euclidean transformation of the plane and, if the set A constitutes a line segment, $St(A)$ is twice the length of this segment.

This paper proves, as indicated by Steinhaus, that Steinhaus measure is a generalization of curve length. Steinhaus believed that, for any set A, $St(A)=2LA$, but such is not the case. The facts are that $St(A)=LR$, $St(I)=0$, where $I=A-R$ and R is the set of those points p of A for which (with $c(p, r)$ the set of points whose distance from p is r)

$$\lim_{r\to 0}\frac{LAc(p,r)}{2r}=1.$$

Let C be a continuous rectifiable curve parametrized according to arc length and, for $i=1, 2, 3, \cdots$, let E_i be the set of points of C of multiplicity i. The author used Steinhaus measure in proving the length of C to be $\sum_{i=1}^{\infty}iL(E_i)$, but believes that a more direct proof is possible. *J. F. Randolph* (Ithaca, N. Y.).

Gleyzal, André. Interval-functions. Duke Math. J. **8**, 223–230 (1941). [MF 4587]

The author considers a set T of open intervals I, a real-valued function φ defined on T and the class K of all points x which are covered by intervals of T of arbitrarily small length. If x is in K, the set $\varphi(x)$ is by definition the set of all numbers η such that there is a sequence I_n of intervals in T containing x, converging in length to zero and for which $\eta=\lim\varphi(I_n)$. This set $\varphi(x)$ is called the kernel of $\varphi(I)$. If the function $\varphi(x)$ is single-valued, the interval-function $\varphi(I)$ is said to be convergent. The author proceeds to give a new proof of the Baire theorem by showing first that a necessary and sufficient condition for a point-function to be the kernel of a convergent interval-function is that it be the limit of a sequence of continuous functions, and, second, that the kernel of a convergent interval-function has a point of continuity interior to every perfect set with

respect to the set and, conversely, if a function has this continuity property it is the kernel of a convergent interval-function. *H. H. Goldstine* (Ann Arbor, Mich.).

Cotlar, Mischa. On non-measurable sets and a generalization of the Lebesgue integral. Publ. Inst. Mat. Univ. Nac. Litoral **2**, 149–176 (1940). (Spanish) [MF 4232]

The author introduces extensions of Lebesgue measure and integration. The pseudo-measure of E is the mean of its inner and outer Lebesgue measures. This definition is introduced indirectly through the notion of a nucleus. A nonempty nucleus is a set of positive outer measure and zero inner measure. Every set is the union of a measurable set and a nucleus, which is determined up to a set of measure zero. The conjugate N^* of a nucleus N is defined up to a set of measure zero as the nucleus of the complement of N. Two nuclei N, N' are nonrampant if N is disjoint from N' and N^* is disjoint from N'^*. Any two sets are nonrampant if they are disjoint and their nuclei are nonrampant. The author proves that the pseudo-measure of a finite or enumerable union of nonrampant sets is the sum of their pseudo-measures. Thus he can extend the whole theory of the Lebesgue integral. *J. W. Tukey* (Princeton, N. J.).

Levi, Beppo. A theory of the Lebesgue integral independent of the notion of measure. Publ. Inst. Mat. Univ. Nac. Litoral **3**, 65–116 (1941). (Spanish. French summary) [MF 5964]

This is the first systematic and detailed exposition of the author's theory of the Lebesgue integral, the fundamental ideas of which he had previously sketched [Ann. Mat. Pura Appl. (4) **1**, 57–82 (1923)]. Let $f(x)$ be a real, nonnegative bounded function in the range $a \leqq x \leqq b$. The starting point is the following definition of the superior integral $\overline{\int_a^b} f(x)dx$. Let $\{(i)\}$ be a set of intervals covering $[a, b]$ in the sense that each x of $[a, b]$ is interior to some interval (i). Now to each interval (i) we associate an altitude $h \geqq 0$ such that to each x in $[a, b]$ there corresponds an (i), containing x, such that the altitude h, associated to (i), satisfies the inequality $h \geqq f(x)$. In short: the set of rectangles of bases (i) and corresponding altitudes h should cover the set of ordinates of $f(x)$. Then we define

$$\overline{\int_a^b} f(x)dx = \text{g.l.b.} \sum ih.$$

The superior integral thus appears as a natural refinement of the superior integral of Riemann and is in fact the equivalent of the exterior measure of the set of ordinates of $f(x)$ in Carathéodory's theory. An extension of $\overline{\int}$ to functions $f(x)$ of both signs allows one to define (in terms of $\overline{\int}$) the inferior integral, integrable functions, measurable sets, measurable functions, until finally Lebesgue's definition of the integral is shown to lead to the author's integral. The paper concludes with an extension of the integral to unbounded functions and to functions of several variables. *I. J. Schoenberg* (Philadelphia, Pa.).

Mammana, Gabriele. Sopra taluni teoremi di inversione. Anais Acad. Brasil. Sci. **13**, 215–232 (1941). [MF 6587]

Proofs are given for the two following theorems concerning Jordan measure and Riemann integration. (1) Let $\{E_n\}$ be a sequence of uniformly bounded Jordan measurable sets for which there exists $E = \lim E_n$. Then, there exists $\lim mE_n$ and $m_iE \leqq \lim mE_n \leqq m_eE$. (2) Let the sequence of functions $\{f_n(P)\}$, defined on a Jordan measurable set G, be uniformly bounded, Riemann integrable and convergent to $f(P)$ on G. Then there exists $\lim \int_G f_n(P)dP$ and $\underline{\int}_G f(P)dP \leqq \lim \int_G f_n(P)dP \leqq \overline{\int}_G f(P)dP$. The author, who was partly motivated by a desire to find an elementary proof of theorem (2), a generalization of Arzela's theorem, does not refer to the similar proof of Arzela's theorem by Landau [Math. Z. **2**, 350–351 (1918)] or to the very simple proof of theorem (2) by Hausdorff [Math. Z. **26**, 135–137 (1927)]. *J. V. Wehausen* (Columbia, Mo.).

Dickinson, D. R. Approximative Riemann-sums for improper integrals. Quart. J. Math., Oxford Ser. **12**, 176–183 (1941). [MF 5984]

Let $f(x)$ be decreasing on (a, b) and $f(x) \to \infty$ as $x \to a^+$. Let $\int_{a+\delta}^b f(x)dx \to I$ as $\delta \to 0$; I is then the Cauchy integral of f on (a, b). The author studies conditions under which the Riemann sum $\sum f(\xi_r)(x_{r+1} - x_r)$ tends to I. Stated descriptively one condition is: (x_0, x_1) not too small in comparison with η, the maximum of $x_{r+1} - x_r$, and ξ_0 not too close to x_0. Another is that ξ_r shall not be too near to x_r. Stated precisely the first condition is: If $0 < \lambda \leqq 1$, $0 < \mu \leqq 1$ then (i) $x_1 - x_0 > \lambda\eta$, and (ii) $\xi_0 - x_0 \geqq \mu(x_1 - x_0)$. The second is: If $0 < \mu \leqq 1$ then $\xi_r - x_r \geqq \mu(x_{r+1} - x_r)$. Another result is: If I exists for $f(x)$ and I_1 for $f_1(x)$, if $f_1(x) \geqq 0$ and decreasing and $|f(x)| \leqq f_1(x)$, then, if $\sum f_1(\xi_r)(x_{r+1} - x_r)$ tends to I_1, $\sum f(\xi_r)(x_{r+1} - x_r)$ tends to I, where the subdivision and choice of ξ_r in the second sum is the same as that for the first. A function $\varphi(x)$ is constructed with $\varphi(x) \geqq 0$ for which I exists, but for which $\sum \varphi(\xi_r)(x_{r+1} - x_r)$ does not tend to I for a subdivision into equal intervals with ξ_r the mid-point of the interval. Conditions are studied for functions on an infinite range. *R. L. Jeffery* (Wolfville, N. S.).

Jeffery, R. L. and Macphail, M. S. Non-absolutely convergent integrals. Trans. Roy. Soc. Canada. Sect. III. (3) **35**, 41–58 (1941). [MF 6023]

A set function $F(e)$ defined and additive on the measurable sets e of an interval (a, b) is absolutely additive with respect to a closed set E if it is absolutely additive over E and $F(\sum c_i) = \sum F(c_i)$, where c_i are the intervals complementary to E. Moreover, $F(e)$ is generalized absolutely additive on (a, b) if there exists a sequence of closed sets E_n such that $(a, b) = \sum E_n$ and $F(e)$ is absolutely additive with respect to each E_n. A measurable function $f(x)$ then has a nonabsolutely convergent integral if there exists a sequence of functions $s_n(x)$ such that, if $F(e) = \lim_n \int_e s_n(x)dx$ (provided this limit exists), then $F(e)$ is generalized absolutely additive on (a, b) and $F(a, x)$, which is the desired integral up to an additive constant, is continuous in x. When $F(e)$ is absolutely additive on E, $f(x)$ is summable on E and $F(e) = \int_e f(x)dx$. The integral is unique up to an additive constant, has an approximate derivative which agrees with $f(x)$ almost everywhere. Conversely, if $F(x)$ has a finite derivative, then the integral of $F'(x)$ defined above reproduces $F(x) - F(a)$. The integral is shown to agree with the general Denjoy integral, the latter being defined as a function $F(x)$ which is generalized absolutely continuous for which $F'_{app}(x)$ exists and agrees with $f(x)$ almost everywhere. Extension is made to a Stieltjes integral, first with respect to a monotonic function $\varphi(x)$ which yields results parallel to the simple case where $\varphi(x) = x$, measure and

derivatives being taken with respect to $\varphi(x)$. The general case where $\alpha(x)$ is any function of bounded variation is reduced to the preceding by the introduction of the total variation φ of α and the function $g(x)=D_\varphi\alpha$ for $D_\varphi(\alpha)=\pm 1$ and $g(x)=0$ otherwise and defining $\int_a^x f(x)d\alpha=\int_a^x f(x)g(x)d\varphi$.

T. H. Hildebrandt (Ann Arbor, Mich.).

Sargent, W. L. C. A descriptive definition of Cesàro-Perron integrals. Proc. London Math. Soc. (2) **47**, 212–247 (1941). [MF 5403]

In this paper the author arrives at a descriptive definition of a Cesàro-Denjoy integral which is equivalent to the C_λ-P integral of Burkill. It is assumed that the C_0-D integral is the C_0-P integral or special Denjoy integral. In defining the C_λ-D integral, λ a positive integer, it is assumed that the C_r-D integral exists for $r=1, 2, \cdots, \lambda-1$, and has the following properties: $F(x)$ is the C_r-D integral of $f(x)$ on (a, x) if (1) $F(x)$ is C_r-continuous; (2) $C_rDF(x)=f(x)$ almost everywhere; (3) the closed interval (a, b) can be covered by a denumerable sequence of closed sets Q_n over each of which F is absolutely continuous and $f(x)$ is summable, and if ω_i^n, μ_i^n are the upper bounds of $|C_r(F, a_i^n, x)-F(a_i^n)|$, $|C_r(F, x, b_i^n)-F(b_i^n)|$, $a_i^n\leqq x\leqq b_i^n$, then $\sum\omega_i^n$, $\sum\mu_i^n$ converge. If Q is a closed set, (a_i, b_i) the intervals complementary to Q, f summable over Q, and C_r-D integral of f on $(a_i, b_i)=F_i(x)$, $a_i\leqq x\leqq b_i$, $\sum_i \overline{\text{bound}}\,|C_r(F_i, a_i, x)|$ and $\sum_i \overline{\text{bound}}\,|C_r(F, x, b_i)-F(b_i)|$ converge, then f is integrable C_r-D in (a, b) and

$$C_r\text{-}D\int_a^b f(t)dt=\int_Q f dt+\sum_i C_r\text{-}D\int_{a_i}^{b_i} f dt.$$

The C_r-D integral is equivalent to the C_r-P integral. A function $F(x)$ is $AC^*(C_\lambda$-sense) (absolutely continuous in the C_λ sense) over a set E if, for every finite set of non-overlapping intervals (a_i, b_i) with a_i, b_i points of E, $\sum\omega_i$, $\sum\mu_i$ are arbitrarily small provided $\sum(b_i-a_i)$ is sufficiently small. $F(x)$ is ACG^* (generalized absolutely continuous in the C_λ sense) if (a, b) can be covered by a sequence of sets over each of which F is $AC^*(C_\lambda$-sense). If $F(x)$ is AC^* over a measurable set E then $C_\lambda DF$ exists, is finite almost everywhere on E, is equal to ADF almost everywhere and is summable over every measurable subset of E. The function $F(x)$ is said to be integrable in the special Cesàro-Denjoy sense of order λ on (a, b) if there is a function $F(x)$ $ACG^*(C_\lambda$-sense) over the closed interval (a, b) and such that $C_\lambda DF=f$ almost everywhere. It is proved that the C_λ-D integral and the C_λ-P integral are equivalent.

R. L. Jeffery (Wolfville, N. S.).

Sargent, W. L. C. On sufficient conditions for a function integrable in the Cesàro-Perron sense to be monotonic. Quart. J. Math., Oxford Ser. **12**, 148–153 (1941). [MF 5980]

The main result of this paper is: If $f(x)$ is integrable in the $C_{\lambda-1}$-P sense, λ a positive integer, if

$$C_\lambda\text{-}\underline{\lim}_{h\to 0+} f(x-h)\leqq f(x)\leqq C_\lambda\text{-}\overline{\lim}_{h\to 0+} f(x+h), \qquad C_\lambda D^+ f\geqq 0,$$

the latter holding except for a denumerable set, then $f(x)$ increases on $a\leqq x\leqq b$. It has been proved by Burkill [Proc. London Math. Soc. (2) **34**, 314–322 (1932)] that, if $f(x)$ is C_λ-continuous and the lower Cesàro derivative $C_\lambda D_* f$ is not less than zero, then $f(x)$ is nondecreasing. This is included in the result given above, without the restriction of C_λ-continuity. There is thus provided a simple and direct proof for a theorem which is fundamental in the development of the Cesàro-Perron scale of integration. *R. L. Jeffery.*

Gunther, N. Remarque à propos des intégrales de Mr. Hellinger. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 99–102 (1941). [MF 4260]

The author generalizes a theorem which he had proved formerly [C. R. (Doklady) Acad. Sci. URSS (N.S.) **21**, 219–223 (1938)]. He considers a domain Ω of points x in an n-dimensional space and infinitely many domains Ω_s $(s=1, 2, \cdots)$ of finite measure, contained in Ω, which approximate Ω for $s\to\infty$; each Ω_s is divided into a finite number of partial domains ω_k $(k=1, \cdots, n)$. Let $\phi(\omega)$, $u(\omega)$ be functions of domains defined for any part of each Ω_s, such that $u(\omega)$ is positive, additive and of bounded variation in every Ω_s and such that the sums

$$S_{sn}=\sum_{k=1}^{n}\frac{\phi(\omega_k)^2}{u(\omega_k)}\omega_k$$

have a finite boundary for all subdivisions of Ω_s and for all s. Then (i) $\phi(\omega)$ can be expressed by a function $\phi(x)$ on Ω which is measurable (B) on Ω as an integral of Stieltjes-Radon type $\phi(\omega)=\omega^{-1}\int_{(\omega)}u(\omega)\phi(x)d\omega$; (ii) the limiting values $\lim_{s\to\infty}\int_{(\Omega_s)}u(\omega)\phi(x)^2d\omega$ and $\lim_{s\to\infty}(\lim_{n\to\infty}S_{sn})$ exist and are equal, thus defining an integral

$$\int_{(\Omega)}\frac{\phi(\omega)^2}{u(\omega)}d\omega,$$

which is a generalization of the notions of Hellinger and Radon. The formulations and the proof are somewhat sketchy and use numerous earlier investigations of the author. *E. Hellinger* (Evanston, Ill.).

Schwartz, H. M. Sequences of Stieltjes integrals. Bull. Amer. Math. Soc. **47**, 947–955 (1941). [MF 5943]

The convergence of Riemann-Stieltjes integrals studied is that of $\int_a^b f dg_n$ to $\int_a^b f dg$, where g_n and g are of bounded variation on $a\leqq x\leqq b$ and $\lim g_n(x)=g(x)$ for all $a\leqq x\leqq b$. The principal new result is the following: If f has at most discontinuities of the first kind, g_n are uniformly of bounded variation, and $\int f dg_n$ and $\int f dg$ exist, then $\int f dg_n\to\int f dg$. The same convergence also holds under a redistribution of the conditions, namely, if $\int f dg_n$ and $\int f dg$ exist, $g_n\to g$ for all x and $V_a^x g_n\to V_a^x g$ for all x. The proof of this result depends essentially on the fact that in this case the approximating sums $\sum f(\xi_i)\Delta_i g_n$ converge uniformly to $\int f dg_n$ in n. The convergence condition $V_a^x g_n\to V_a^x g$ can be weakened to the existence of $\int f dv$ for all functions $v(x)$ which are limits of the sequence $V_a^x g_n$. An extension to the infinite interval (a, ∞) is stated. *T. H. Hildebrandt.*

Erim, Kerim. Über eine neue Definition des mehrdimensionalen Stieltjesschen Integrals. II. Rev. Fac. Sci. Univ. Istanbul (A) **6**, 12–17 (1941). (German. Turkish summary) [MF 6179]

The author discusses further his generalization to functions of several variables of A. H. Copeland's definition of the Riemann-Stieltjes integral [cf. Bull. Amer. Math. Soc. **43**, 581–588 (1937); Rev. Fac. Sci. Univ. Istanbul **4**, 167–182 (1939); these Rev. **1**, 208]. The chief point established here is that, as in the one-dimensional case, the new definition has a domain of applicability actually more inclusive than the classical definition. *J. A. Clarkson.*

Menger, Karl. Analysis and metric geometry. Line integrals, their semicontinuity properties, and their independence of the path. Rice Inst. Pamphlet **27**, 1–40 (1940). [MF 6080]

In this, the first of "Three Lectures on Mathematical Subjects" delivered by the author at the Rice Institute in 1939, the author applies metric methods to four fundamental problems concerning line integrals $J(C)$. They are (1) the existence of $J(C)$, (2) the semicontinuity of $J(C)$ as a functional of C, (3) the existence of a minimizing curve and (4) the independence of path. In (1) one seeks conditions on the integrand F and the curve C, in (2) conditions on F, in (3) conditions on the class of curves $\{C\}$ and the integrand F and in (4) conditions on F. The author prefaces a sketch of his results by a brief but illuminating review of the classical treatment of these problems on the basis of (1) Riemann integration and (2) Lebesgue integration. In a critique of these earlier results, Carathéodory's integrand is mentioned as leading to a type of problem which the classical theories, with their restrictions to curves of finite lengths, exclude from consideration. A second example exhibits a line integral $\int udx+vdy$ which is independent of the path but since neither u_y nor v_x exist Cauchy's classical condition cannot be applied. Consideration of a third procedure in which the integral is the limit of Weierstrass sums points the way to the introduction of metric methods. Identifying such a sum as the length of a polygon inscribed in C, where the determination of the length is based upon the F-distance $pq=F(p, \vartheta_{pq})\cdot\overline{pq}$ of two points p, q, where $\overline{pq}$ is the Euclidean distance and ϑ_{pq} denotes the angle between the vector from p to q and the positive x-axis, the limit of such a sum (if it exists) is the F-length of the curve C and is the functional the author associates with C and for which the problems of existence, semicontinuity and minimizing curves are considered. The procedure is to develop a theory of arc length in such an F-metric space. As a result, the author solves these problems under conditions considerably weaker than those heretofore imposed and obtains a four-fold generalization of these problems in which the integrand, class of admissible curves, functional of comparison and the underlying space are all extended. The reader is referred to the paper for the details as well as for the necessary and sufficient condition for independence of path with which the lecture closes. Since the lecture is a review of work which the author has published over a period of years, no proofs are given.

L. M. Blumenthal (Columbia, Mo.).

Menger, Karl. On shortest polygonal approximations to a curve. Rep. Math. Colloquium (2) **2**, 33–38 (1940). [MF 5773]

In a convex and compact metric space designate, for any finite point set $F=(p_1, \cdots, p_n)$, by $\lambda(F)$ the minimum of $\sum_{i=1}^{n-1} p_{r_i}p_{r_{i+1}}$, where $(r_1, \cdots, r_n)$ traverses all permutations of $(1, \cdots, n)$. Since the length $L(\overline{pq})$ of a segment $\overline{pq}$ equals pq, the number $\lambda(F)$ is the length of the shortest polygonal line inscribed to F. Let the continuous curve C be a shortest path through all its points, that is, $L(C')\geqq L(C)$ for any curve C' whose basic set contains the points of C. Then $\underline{\lim}\,\lambda(F_i)\geqq\lambda(C)$ for any sequence of finite sets F_i which approach C (that is, $\max_{x\in C}r_{p,y\in C}(xC+yF_i)\to 0$). If $\underline{\lim}\,\lambda(F_i)=L(C)<\infty$, then the shortest polygonal lines inscribed to F_i approach C.

H. Busemann (Chicago, Ill.).

Haupt, Otto. Zur Bestimmung des Oberflächenmasses vermittelst geometrisch ausgezeichneter Polyederfolgen. Jber. Deutsch. Math. Verein. **51**, 170–192 (1941).

The author first extends a well-known result of Fréchet for surfaces to k dimensional surfaces in n-space, $n\geqq 3$, $2\leqq k\leqq n-1$. This extension may be stated roughly as follows: Let S be such a surface which can be represented topologically on a cube W in k-space by a vector function $x=x(u)$ of class C' in which there exist numbers m and M with $0<m\leqq M$ such that

$$m\cdot|u_1-u_2|\leqq|x(u_1)-x(u_2)|\leqq M\cdot|u_1-u_2|$$

for all $u_1, u_2\in W$. Let $\{\Pi_p\}$ be a sequence of polyhedra inscribed in S and tending to S such that the ratio of the longest to the shortest side of each simplex s of each Π_p is uniformly bounded (independently of s and p). Then the elementary areas of the Π_p tend to the area of S as defined by the classical integral. Various other theorems are proved, showing how the area of S may be obtained as the limit of the areas of certain simplicial subsets of the Π_p when S is representable by $x=x(u)$, where $x(u)$ still satisfies the above m, M condition but satisfies weaker conditions of differentiability.

C. B. Morrey, Jr. (Berkeley, Calif.).

Radó, Tibor. On the semi-continuity of double integrals in parametric form. Trans. Amer. Math. Soc. **51**, 336–361 (1942). [MF 6319]

The writer considers parametric integrals of the form $I(x)=\int\int_B f(x_1, x_2, x_3, X_1, X_2, X_3)dudv$, in which f is of class C'' in all its arguments for $|X|=(X_1^2+X_2^2+X_3^2)^{\frac{1}{2}}\neq 0$ and is positively homogeneous of the first degree in the X_i. Vector functions $x(u, v)=[x_1(u, v), x_2(u, v), x_3(u, v)]$ are of class K_1 on a closed Jordan region $\bar{B}$ if (1) they are continuous on $\bar{B}$, (2) the partial derivatives $x_{iu}, x_{iv}, i=1, 2, 3$, exist almost everywhere on the interior B, and (3) the Jacobians X_1, X_2, X_3 are summable on B. The class K_2 consists of the subset of K_1 such that there exists a sequence $\{x_n\}$ of vector functions which are quasi-linear on polygonal regions B_n and such that the surfaces S_n tend in the sense of Fréchet to S with $L(S_n)\to L(S)$ (Lebesgue area). Without further hypotheses on f, it is shown that, if an oriented surface S possesses two representations $x=x^{(1)}(u, v)$ and $x=x^{(2)}(u, v)$ of class K_2, then $I(x^{(1)})=I(x^{(2)})$. Let $S_n: x=x_n(u, v)$, and $S_0: x=x_0(u, v)$, x_n and $x_0\in K_2$, and suppose $S_n\to S$ in the sense of Fréchet. Then various lower semicontinuity theorems are proved such as the following: if (1) $f(x, X)\geqq 0$ for all x near the point set covered by S_0 and if (2) $f(x_0, X_0)$ is regular at almost all points of B_0, then $I(x_0, B_0)\leqq\liminf_{n\to\infty} I(x_n, B_n)$. If (1) holds and $f(x_0, X_0)$ is quasi-regular almost everywhere on B_0, the conclusion above holds at least if $L(S_n)$ is uniformly bounded.

C. B. Morrey, Jr.

Jones, F. B. Connected and disconnected plane sets and the functional equation $f(x)+f(y)=f(x+y)$. Bull. Amer. Math. Soc. **48**, 115–120 (1942). [MF 6188]

By an argument similar to that employed by Hamel [Math. Ann. **60**, 459–462 (1905)] in demonstrating the existence of discontinuous solutions of the functional equation $f(x+y)=f(x)+f(y)$, the author shows the existence of discontinuous solutions whose graphs are connected in the topological sense. Using such graphs, very simple existence proofs are obtained for most of those "pathological" noncompact planar point sets [biconnected sets, connected +locally connected+punctiform sets, etc.] that were con-

tributed especially during the 1920's [vide early volumes of Fundamenta Mathematicae, for example].

R. L. Wilder (Ann Arbor, Mich.).

Besicovitch, A. S. Remark on relative derivatives. J. London Math. Soc. **16**, 210–211 (1941). [MF 6614]

The author gives a very simple and elementary proof of the known theorem to the effect that, if $x(t)$ and $y(t)$ are continuous in (a, b) and

$$\lim_{h\to 0} \{y(t+h)-y(t)\}/\{x(t+h)-x(t)\}=0$$

for all t, then $y(t)$ is constant. *W. Feller.*

Balanzat, Manuel. On a limiting function of continuous functions. Revista Union Mat. Argentina **7**, 140–143 (1941). (Spanish) [MF 6358]

Ridder, J. Über approximative Differentiation von Reihen. Nieuw Arch. Wiskde (2) **20**, 301–306 (1940). [MF 5215]

Let the functions $f^{(n)}(x)$ have one-sided approximate derivatives $f^{(n)}_{appr;1}(x)$ and $f^{(n)}_{appr;r}(x)$ and suppose that $f^{(n)}_{appr;1}(x) \leqq f^{(n)}_{appr;r}(x)$ in $[a, b]$. Then, if the two series $\phi(x)=\sum f^{(n)}_{appr;1}(x)$ and $\psi(x)=\sum f^{(n)}_{appr;r}(x)$ converge uniformly in $[a, b]$, and if $\sum f^{(n)}(x_0)$ converges for some point x_0, then $\sum f^{(n)}(x)$ converges uniformly to a function $F(x)$ such that $F'_{appr;1}(x)=\phi(x)$ and $F'_{appr;r}(x)=\psi(x)$. The analogous theorem holds for ordinary one-sided derivatives. *W. Feller.*

Birindelli, Carlo. Sopra un teorema di derivazione per serie del Tonelli. Ann. Scuola Norm. Super. Pisa (2) **10**, 157–165 (1941). [MF 6422]

The point of departure is a paper by Tonelli [Atti Accad. Naz. Lincei. Rend. (6) **13**, 163–168 (1931)] in which the convergence of $\sum_m f_m^{(k)}(x)$ is proved assuming that $f_m^{(k)}(x)$ is monotone in the same sense for all m and that the series $\sum f_m(x)$ converges at $k+2$ points. The author assumes instead that $f_m^{(k-1)}(x)$ is convex in the same sense for all m and replaces convergence by summability B_m, where $B_m=\sum_0^m b_{mn}u_n$ is the transform of $\sum_0^\infty u_n$. Here the $b_{mn}>0$ and $b_{mn}\uparrow 1$ when $m\to\infty$, n being fixed. If $\sum f_m(x)$ is summable B_m at $k+2$ points, then it is uniformly summable B_m in the least interval containing these points, and the derived series $\sum f_m^{(r)}(x)$ are also summable B_m in any interior interval for $r=1, 2, \cdots, k-1, k$, the kth derived series possibly only everywhere, the sum of the series being the derivatives of the sum. Extension to several variables: if $f_n(x, y)$ is real and convex in a convex domain and the series $\sum f_n(x, y)$ is summable B_m at four points, no three of which are collinear, then the series is summable B_m uniformly in the least triangle containing the points, and the first order derived series are also summable at interior points where the terms have partial derivatives, the sum of the derived series being the partial derivatives of the sum of the original series. *E. Hille* (New Haven, Conn.).

Cesari, Lamberto. Sulle funzioni assolutamente continue in due variabili. Ann. Scuola Norm. Super. Pisa (2) **10**, 91–101 (1941). [MF 6415]

A continuous function $f(x, y)$ defined in the square S, $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, is said to be absolutely continuous in the Tonelli sense over S if (a) for almost every value of x the function f is an absolutely continuous function of y, (b) the total variation of f with respect to y is a summable function of x, (c) the above conditions remain satisfied when the roles of x and y are interchanged. These hypotheses imply in particular that the partial derivatives f_x and f_y exist almost everywhere in S and are summable over S. It was shown by Saks [Ann. of Math. (2) **34**, 114–124 (1933)] that a function absolutely continuous in Tonelli's sense need not have a Stoltz differential at a single point. The author now shows that, if $f(x, y)$ is absolutely continuous in Tonelli's sense over S, and in addition $|f_x|^{2+\alpha}$ and $|f_y|^{2+\alpha}$ are summable over S for some $\alpha>0$, then $f(x, y)$ has a Stoltz differential at almost every point. Moreover, it is shown that the theorem does not hold for $\alpha=0$; as in the Saks example, the function f may then be without Stoltz differential at all points. *A. Zygmund.*

Fourier Series and Generalizations, Integral Transforms

* **Jackson, Dunham. Fourier Series and Orthogonal Polynomials.** Carus Monograph Series, no. 6. Mathematical Association of America, Oberlin, Ohio, 1941. xii+234 pp. $2.00. [$1.25 to members. Non-members order from Open Court Publishing Co., La Salle, Ill.]

The term "Fourier series" is to be taken as including series of orthogonal functions in general, regarded as expansions associated with a given function. The book is intended for a student who knows a reasonable amount of calculus and a little about differential equations, and in addition has a certain amount of "mathematical maturity." For such a student, it could serve as an introduction to the methods of more advanced analysis as well as to the subject matter itself. As the author states, "Under the circumstances, 'rigor' in the sense of literal completeness of statement has been out of the question. It is hoped, however, that the reader who is familiar with the methods of rigorous analysis will be able without any difficulty to read between the lines the requisite supplementary specifications, and will find that what has actually been said is entirely accurate in the light of such interpretation." The author has been able, by not proving everything, to include a great deal of interesting material; by proving typical results in detail, he has shown how the formal results may be justified and safely used. Considerable attention is paid to physical applications of orthogonal functions. This rather unusual synthesis of two different attitudes should be good for students beginning advanced work in either pure or applied analysis.

A summary by chapters follows. I. Fourier series (in the original sense). Riemann's theorem; Parseval's theorem; some sufficient conditions for convergence (functions of bounded variation are not mentioned, but connections with the theory of approximation are stressed, and Lebesgue's method is used to prove uniform convergence for a function satisfying a Lipschitz condition); summability by arithmetic means. II. Legendre polynomials. Definition by, and properties derived from, the generating function; Laplace's integral representation and bounds derived from it; sufficient conditions for convergence. III. Bessel functions. Definition by the differential equation; existence of zeros; orthogonality; no convergence theory. IV. Boundary value problems leading to Fourier, Legendre and Bessel series. V. Double series; Laplace series. Boundary value problems leading to them, and a discussion of spherical harmonics. VI. The Pearson frequency functions. These are included

because of their interest as weight functions for orthogonal polynomials. VII. Orthogonal polynomials. Orthogonalization with respect to a weight function; general properties, including zeros and least square property. VIII, IX, X. Jacobi, Hermite and Laguerre polynomials. Special properties of each kind; application of the last two to simple wave equations. XI. Convergence. Sufficient conditions in the case of uniformly bounded sets of orthonormal polynomials; conditions on the weight function so that a set will be uniformly bounded. A collection of exercises at the end of the book provides illustrations and extensions of the material. There is a short bibliography.

R. P. Boas, Jr. (Chapel Hill, N. C.).

Perron, Oskar. Über die Approximation stetiger Funktionen durch trigonometrische Polynome. Math. Z. **47**, 57–65 (1940). [MF 6297]

An exposition of the principle of approximation by singular integrals [see, for example, Lebesgue, Ann. Fac. Sci. Univ. Toulouse (3) **1**, 25–117 (1909)] for cases in which the kernel as a function of t and x is a trigonometric sum in $t-x$, and the approximating functions consequently are trigonometric sums. Attention is given to uniform convergence of the approximation to a continuous function, to interpretation of the process as a method of summation of Fourier series, to convergence at a point of discontinuity and to the extension to functions of several variables.

D. Jackson (Minneapolis, Minn.).

Schubert, Hans. Ein potentialtheoretischer Hilfssatz. Math. Z. **47**, 8–15 (1940). [MF 6293]

This paper localizes to an interval $\theta_1 \leqq \theta \leqq \theta_2$ the well-known theorem of Privaloff [Bull. Soc. Math. France **44**, 100–103 (1916)] that if a function $f(\theta)$ with period 2π satisfies a Lipschitz condition of order α, $0<\alpha<1$, then so does its conjugate function. [The relatively long proof of the author is unnecessary, the localization following immediately from the Privaloff result. See, for example, J. L. Walsh and W. E. Sewell, Duke Math. J. **6**, 658–705 (1940), in particular, p. 668, footnote; these Rev. **2**, 80.]

L. H. Loomis (Cambridge, Mass.).

Weissinger, Johannes. Ein Satz über Fourierreihen und seine Anwendung auf die Tragflügeltheorie. Math. Z. **47**, 16–33 (1940). [MF 6294]

A very well-known theorem of Privaloff [Bull. Soc. Math. France **44**, 100–103 (1916)] asserts that, if a function $f(x)$ of period 2π satisfies a Lipschitz condition of order α, $0<\alpha<1$, then the conjugate function

$$\bar{f}(x)=\pi^{-1}\int_{-\pi}^{\pi} f(x+t)\tfrac{1}{2}\operatorname{cotg}\tfrac{1}{2}t\,dt=\pi^{-1}\lim_{\delta\to+0}\left[\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right]$$

also satisfies a Lipschitz condition of order α. In the present paper the author generalizes this theorem by assuming that (1) $f(x)$ is absolutely integrable; (2) $f(x)$ has at most a finite number $x_1, x_2, \cdots, x_k$ of discontinuities; (3) in every closed interval I not containing any of the points x_j the function f satisfies a Lipschitz condition of order α. The conclusion is that in every such interval I the function $f(x)$ satisfies a Lipschitz condition of order α. It may be added that the theorem can also be deduced directly from Privaloff's result by observing that, if two integrable functions f_1 and f_2 coincide in an interval (a, b), the difference $\bar{f}_1(x)-\bar{f}_2(x)$ is regular for $a<x<b$. *A. Zygmund.*

Wang, Fu Traing. Note on the absolute summability of Fourier series. J. London Math. Soc. **16**, 174–176 (1941). [MF 6015]

If $\sum(a_n^2+b_n^2)(\log n)^{1+\epsilon}<\infty$, $\epsilon>0$, and if α is any number greater than $\frac{1}{2}$, the series

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

is summable $|C, \alpha|$ for almost every x. *A. Zygmund.*

de Oliveira Castro, F. M. On the representation of analytic functions by Fourier integrals. Anais Acad. Brasil. Sci. **12**, 313–315 (1940). (Portuguese) [MF 6577]

The author observes that a function $f(z)$ belonging to L on the real axis, analytic in $y\geqq 0$ and such that for each z_0 in $y\geqq 0$

$$\lim_{\rho\to\infty}\int_{C(\rho)} f(z)(z-z_0)^{-1}dz=0,$$

where $C(\rho)$ is a semicircle with center at the origin and radius ρ, can be represented in $y\geqq 0$ by a Fourier integral.

R. P. Boas, Jr. (Chapel Hill, N. C.).

Dieulefait, C. E. On Thiele's semi-invariants and the conjugate Fourier function. An. Soc. Ci. Argentina **129**, 208–211 (1940). (Spanish) [MF 3776]

The inverse Fourier transform formula is obtained formally by a method which appears justifiable only under very restrictive conditions not stated by the author.

R. P. Boas, Jr. (Chapel Hill, N. C.).

Edmonds, Sheila M. On the Parseval formulae for Fourier transforms. Proc. Cambridge Philos. Soc. **38**, 1–19 (1942). [MF 6029]

The paper gives a number of conditions for the validity of the formulas

$$(*)\qquad \int_0^{\infty} F_c(x)G_c(x)dx=\int_0^{\infty} f(x)g(x)dx,$$

$$(**)\qquad \int_0^{\infty} F_s(x)G_s(x)dx=\int_0^{\infty} f(x)g(x)dx,$$

where F_c and F_s denote respectively the cosine and the sine transforms of the function f, and similarly for G_c, G_s. The transforms are defined by ordinary convergence. (The problem here is different from the apparently similar one concerning, for example, the validity of the formula $\int_0^{\infty}F_c(x)g(x)dx=\int_0^{\infty}f(x)G_c(x)dx$). The following two theorems may serve as samples of the results obtained. (i) Suppose that $g(x)\in L$ in any finite interval, tends to 0 as x approaches infinity and is of bounded variation in some interval (a, ∞), and that $f(x)\in L$ and is of bounded variation in the neighborhood of every point where $g(x)$ is not. Then the formulas (*) and (**) hold, provided the integrals on the left are meant as $\lim\int_{\epsilon}^{\omega}$ for $\epsilon\to 0$, $\omega\to+\infty$. (ii) If $f(x)$ belongs to $L(0, \infty)$ and is bounded in the neighborhood of the origin, and $g(x)$ belongs to $L(0, 1)$, is nonincreasing and tends to 0 with $1/x$, then

$$\lim_{\epsilon\to+0}\int_{\epsilon}^{1} F_c(x)G_c(x)dx + \lim_{X\to+\infty}\int_1^X\left(1-\frac{x}{X}\right)F_c(x)G_c(x)dx=\int_0^{\infty} f(t)g(t)dt,$$

and similarly for sine transforms. *A. Zygmund.*

Wintner, Aurel. On a family of Fourier transforms. Bull. Amer. Math. Soc. **48**, 304–308 (1942). [MF 6410]

Taking $\mu \geqq 1$ and $\lambda > 0$ the author considers those combinations (λ, μ) for which the Fourier transform of the function $f(t)$ which equals $(1-|t|^\lambda)^\mu$ for $|t| \leqq 1$, and zero for $|t| \geqq 1$, is nonnegative. The author argues that the limiting case $\mu = \infty$ corresponds essentially to the function $f(t) = e^{-|t|^\lambda}$, and in this case it is known that the transform is nonnegative if $\lambda \leqq 2$, but fails to be nonnegative for every $\lambda > 2$. In contrast to this he proves that, for every finite μ, the transform is nonnegative if $\lambda \leqq 1$, and fails to be nonnegative for every $\lambda > 1$. *S. Bochner* (Princeton, N. J.).

Wintner, Aurel. The singularities of Cauchy's distributions. Duke Math. J. **8**, 678–681 (1941). [MF 5951]

The author discusses the analytic continuation $f_\lambda(z)$ of the density function of the stable symmetric distribution with exponent λ, using the characteristic function $e^{-|t|^\lambda}$. If $\lambda > 1$, $f_\lambda(z)$ is a transcendental entire function; if $\lambda = 1$, $f_\lambda(z) = 1/\pi(1+z^2)$; if $\lambda < 1$, $f_\lambda(z)$ is an analytic function possessing a branch point at the origin, a transcendental singularity, which is the only finite singularity.

J. L. Doob (Princeton, N. J.).

Schoenberg, I. J. Positive definite functions on spheres. Duke Math. J. **9**, 96–108 (1942). [MF 6343]

If a function of two points on the sphere $S_m : x_1^2 + \cdots + x_m^2 = 1$ depends only on the spherical distance of the two points, then it has the form $f(\cos\theta)$, where θ is taken in $(-\pi, \pi)$. The author proves that such a function is positive-definite only if it is a convergent sum $\sum_{n=0}^\infty a_n P_n^{(\lambda)}(\cos\theta)$, $a_n \geqq 0$, $\lambda = \frac{1}{2}(m-1)$, $P_n^{(\lambda)}(\cos\theta) =$ spherical harmonic. This first theorem of the author has been generalized in the meantime, by the reviewer, from spheres to compact spaces with transitive groups of transformations. However, the author carries out the limit $m \to \infty$, thus obtaining the expansion $f(\cos\theta) = \sum_{n=0}^\infty a_n (\cos\theta)^n$, $a_n \geqq 0$, valid in the (noncompact) Hilbert space S_∞: $\sum_{n=1}^\infty x_n^2 = 1$. He also shows that any distance function on S_∞ which is a continuous function of the original distance has the form $(\sum_{n=0}^\infty a_n(1-(\cos\theta)^n))^{\frac{1}{2}}$, and here again the author has a proof for this special case of a noncompact space.

S. Bochner (Princeton, N. J.).

* **Widder, David Vernon. The Laplace Transform.** Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., 1941. x+406 pp. $6.00.

The present book is a significant contribution to a field of analysis whose importance becomes more and more obvious. Except for a recent book by Doetsch [Theorie und Anwendungen der Laplace-Transformation, Springer, Berlin, 1937], there are no other books on the theory of Laplace transforms. The present book, the material of which overlaps but little with that of Doetsch, will be particularly welcomed by anyone who needs a general introduction to this fascinating subject. It is very clearly written and can be easily understood by a student who has a knowledge of principles of functions of a single real or a single complex variable. At the same time it contains a considerable amount of new material resulting from researches of the author and of R. P. Boas. Chapter 1 (The Stieltjes integral) is an introduction to the theory of the Riemann-Stieltjes integral. Chapter 2 (Fundamental formulas) gives fundamental facts concerning the domain of convergence (simple, absolute, uniform), uniqueness of the determining function, and inversion formulas by means of complex integration. Chapter 3 (The moment problem) contains a complete treatment of monotonic sequences and their applications to the theory of Hausdorff moment problem, and Hausdorff method of summation of sequences. There are also conditions for the existence of solutions of the Stieltjes and Hamburger moment problems. Chapter 4 (Absolutely and completely monotonic functions) gives an excellent exposition of the theory due to S. Bernstein and the author. It also contains a recent result of the author on completely convex functions, which inspired the work of several mathematicians during the last year. In chapter 5 (Tauberian theorems) one finds interesting material on Abelian and Tauberian theorems connected with the theory of Laplace transforms; among the applications various proofs of the prime-number theorem should be mentioned. Chapter 6 contains an exposition of the theory of bilateral Laplace transforms and Mellin transforms. Chapter 7 (Inversion and representation problems for the Laplace transform) gives a treatment of the inversion formula based on the knowledge of derivatives of the generating function for large real values of the independent variable. The author has contributed a great deal to the development of this theory, the discovery of which he attributes to E. L. Post [1930], but which actually was discovered by Stieltjes [Correspondence d'Hermite et de Stieltjes, Vol. 2, Letter 383 (1893)]. The final chapter 8 (The Stieltjes transform) presents an analogous theory for the Stieltjes transforms and their iterations. There are but very few topics of the theory of Laplace transforms which are not treated in the book. Among these should be mentioned some function theoretical applications, in particular to the theory of functions analytic in a half plane, or in a strip. *J. D. Tamarkin* (Providence, R. I.).

Vignaux, J. C. Differentiation and integration of the Laplace transform. An. Soc. Ci. Argentina **129**, 26–31 (1940). (Spanish) [MF 3773]

Some rather obvious remarks on $\int_0^\infty e^{-st}\varphi(t)dt$, $c > 0$, and the corresponding double Laplace integral.

R. P. Boas, Jr. (Durham, N. C.).

Avakumović, Vojislav G. Bemerkungen über Laplacesche Integrale, deren Wachstum von Exponentialcharakter ist. III. Math. Z. **47**, 141–152 (1940). [MF 6305]

The author proves two function-theoretical Tauberian theorems, a typical one of which is: Let n be a positive integer, let $A(u)$ be nondecreasing, let $H(s) \equiv \int_0^\infty e^{-su} dA(u)$ be convergent for $\Re(s) > 0$ and finally let $H(s) - A\exp[s^{1/(2n-1)}]$, $A \neq 0$, be uniformly bounded in a convex domain whose boundary has contact of order $2n-1$ with the imaginary axis at $s=0$. Then

$$A(u) \sim (2n-1)^{(1-2n)/4n} \frac{A}{2(n(2n-1)\pi)^{\frac{1}{2}}} u^{(1-2n)/4n} \times \exp[2n(2n-1)^{(1-2n)/4n} u^{(2n-1)/2n}],$$

as $u \to \infty$. The theorem generalizes an earlier result by the same author contained in paper number II with the above title [Math. Z. **46**, 67–69 (1940); see these Rev. **1**, 228]. The method of proof represents a nontrivial generalization. The author states that the present paper is a special case of a general inversion result previously stated by him but with proof not published. *W. T. Martin.*

Mossin Kottin, Cecilia. On functional expansions of the form $f(x) = \sum c_n x/(x^2+n^2)$. Revista Union Mat. Argentina **7**, 41–44 (1941). (Spanish) [MF 4457]

Let $f(z)$ be analytic in $\Re(z) \geqq \frac{1}{2}$ and at $z = \infty$, with the development $\sum_{n=0}^\infty a_n z^{-1}(1-z^{-1})^n$. Then $f(z)$ is the Laplace

transform of $\varphi(t)=\sum_0^\infty a_m L_m(t)$, where $L_m(t)$ is the mth Laguerre polynomial. If $\varphi(t)=\sum_0^\infty c_n \cos nt$ $(0\leqq t\leqq\pi)$, one then has (formally; rigorously if $\sum|c_n|<\infty$) $f(z)$ expanded in a series of the form mentioned in the title. The c_n's can be expressed in terms of derivatives of $f(z)$ as follows.

$$a_m=\frac{1}{m!}\frac{d^m}{dx^m}[xf(x)]\Big|_{x=1},$$

$$c_n=2\pi^{-1}\sum_{m=0}^{\infty}a_m L_m(I_n),$$

where exponents on I_n are to be interpreted as superscripts and

$$I_k^n=\int_0^\pi t^k\cos nt\,dt.$$

R. P. Boas, Jr. (Chapel Hill, N. C.).

Auluck, F. C. The Fermi-Dirac functions. Philos. Mag. (7) **33**, 159–160 (1942). [MF 6260]

Using the bilateral Laplace transform the author obtains asymptotic expressions for $\int_0^\infty\phi'(x)\{e^{x-t}+1\}^{-1}dx$. The special case $\phi'(x)=x^k$ has recently been treated by Cooper [Philos. Mag. (7) **30**, 187–189 (1940); cf. these Rev. **2**, 96]. *W. Feller* (Providence, R. I.).

Boas, R. P., Jr. Generalized Laplace integrals. Bull. Amer. Math. Soc. **48**, 286–294 (1942). [MF 6408]

The author generalizes the Laplace integral by showing that, if $g(z, t)$ is in a certain sense "close" to e^{-zt}, then for any function $f(z)$, $z=x+iy$, such that

$$\int_{-\infty}^{\infty}|f(x+iy)|^2dy\leqq M,\qquad x>c,$$

there exists a $\psi(t)$ such that $f(z)$ is represented by

$$\int_0^\infty g(z,t)\psi(t)dt,\qquad x>c.$$

The "closeness" criterion is essentially that for any $a(t)\in L^2(S, R)$,

$$\int_{-\infty}^{\infty}\left|\int_S^R a(t)[e^{-(c+iy)t}-g(c+iy,t)]dt\right|^2dy$$
$$<\lambda\int_{-\infty}^{\infty}\left|\int_S^R a(t)e^{-(c+iy)t}dt\right|^2dy,\quad 0<\lambda<1.$$

This "closeness" criterion is very similar to a relationship in a theorem of Wiener and Paley, referred to by the author. In fact the author's result is the Laplace transform analogue of the result of Wiener and Paley, which is for Fourier series. The author gives various sufficient conditions that $g(z, t)$ be "close" to e^{-zt} and in particular shows that a satisfactory $g(z, t)$ is $(2/\pi)^{\frac{1}{2}}K_\nu(zt)(zt)^{\frac{1}{2}}$, $-\frac{1}{2}<\Re\nu<\frac{1}{2}$, where $K_\nu(z)$ has the meaning usual in the theory of Bessel functions. *N. Levinson* (Cambridge, Mass.).

Boas, R. P., Jr. Inversion of a generalized Laplace integral. Proc. Nat. Acad. Sci. U. S. A. **28**, 21–24 (1942). [MF 6007]

The author considers the transform formula

$$f(z)=(2z/\pi)^{\frac{1}{2}}\int_0^\infty t^{\frac{1}{2}}K_\nu(zt)d\alpha(t),$$

in which K_ν denotes a Bessel function. This transform formula is closely related to the Laplace transform, as the author has shown [see the preceding review]. Meyer obtained an inversion formula for the above transform which is a generalization of the well-known complex inversion formula for the Laplace transform. Here the author gives an inversion formula which generalizes the Post-Widder inversion operator for Laplace transforms. *N. Levinson.*

Faedo, Sandro. Sulle trasformate multiple di Laplace. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 722–727 (1941). [MF 5752]

Let $F(t, v)$ be the double Laplace transform of $f(t, T)$ over $a\leqq t\leqq b$, $c\leqq T\leqq d$. Suppose

(1) $$|F(z,v)|<kz^m v^n e^{s(d-b)+r(c-d)},$$

where $a<b-r$, $c<d-s$, z and v are positive real numbers and k, m, n are constants. If f is analytic, then (1) implies $f=0$. If f is summable and $r=s=0$, it appears that $f=0$, though the author's argument is incomplete. The theorem is an extension of the known result for one dimension [cf. Picone, Appunti di Analisi Superiori, Napoli, 1940, p. 718; cf. these Rev. **3**, 144]. *D. G. Bourgin* (Urbana, Ill.).

Kendall, M. G. Corrections to a paper on the uniqueness problem of moments. Ann. Math. Statistics **12**, 464–465 (1941). [MF 6057]

The paper appeared in Ann. Math. Statistics **11**, 402–409 (1940) and was reviewed in these Rev. **2**, 191. *W. Feller* (Providence, R. I.).

Robertson, Fred. Properties of a generalized operator. Iowa State Coll. J. Sci. **16**, 173–188 (1942). [MF 5976]

This paper contains a formal discussion of various properties of a known set of real-order derivative and integral operators. *J. L. Barnes* (Medford, Mass.).

Agnew, Ralph Palmer. Limits of integrals. Duke Math. J. **9**, 10–19 (1942). [MF 6337]

It is first proved that, if

$$L(\lambda)=\lim_{A\to\infty}\int_A^{A+\lambda}f(t)dt$$

exists for each λ in some set having positive measure, then $L(\lambda)$ exists for each real λ and $L(\lambda)=\lambda L$, where $L=L(1)$; moreover, the convergence is uniform over each finite interval $-a\leqq\lambda\leqq a$. Applications of this result are then given, first to the proofs of two theorems of Iyengar [Proc. Cambridge Philos. Soc. **37**, 9–13 (1941); these Rev. **2**, 218], and second to the proof of the Tauberian theorem: if $F(t)$ is absolutely continuous over each finite interval and

$$\lim_{A\to\infty}\int_A^{A+\lambda}[F(t)-F'(t)]dt=0,\quad -\infty<\lambda<\infty,$$

and $\lim_{t\to\infty}e^{-t}F(t)=0$, then $\lim_{t\to\infty}F(t)=0$. Further implications of the existence of the limit $\lim_{A\to\infty}\int_A^{A+\lambda}f(t)dt$ are discussed. *H. L. Garabedian* (Evanston, Ill.).

Garabedian, H. L. A class of linear integral transformations. Amer. J. Math. **64**, 208–214 (1942). [MF 5999]

When $\chi(t)$ is a suitably restricted mass function, the function-to-function transformation

$$(\chi)\qquad y(s)=[\chi(1)-\chi(1-)]x(s)+s^{-1}\int_0^s\chi'(t/s)x(t)dt$$

from $x(t)$ to $y(s)$ provides a method of summability, studied

by Silverman [Trans. Amer. Math. Soc. 26, 101–112 (1924)], under which $x(t)$ is summable to L if $y(s)\to L$ as $s\to\infty$. This is analogous to the Hausdorff sequence-to-sequence transformation

$$[\chi] \qquad y_n=\int_0^1 \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k}s_k\,d\chi(t)$$

from x_k to y_n by which s_k is summable to L if $y_n\to L$ as $n\to\infty$. One says that $(\chi_2)\supset(\chi_1)$ if $x(t)$ is summable (χ_2) to L whenever $x(t)$ is bounded and integrable over each finite interval and is summable (χ_1) to L; and $[\chi_2]\supset[\chi_1]$ if $y_n\to L$ whenever $x_n\to L$. It is shown that so long as all functions involved have certain properties, among which is the property that the derivatives $\chi_i'(t)$ be continuous over $0<t<1$, the equivalent criteria of Hausdorff, Hille, Tamarkin, Garabedian and Wall for $[\chi_2]\supset[\chi_1]$ are sufficient conditions for $(\chi_2)\supset(\chi_1)$. Results of Silverman are used in the proof. *R. P. Agnew* (Ithaca, N. Y.).

Mohan, B. A self-reciprocal function in the form of a series. Bull. Calcutta Math. Soc. **32**, 125–128 (1940). [MF 6163]

Using one of his criteria for self-reciprocal functions [Quart. J. Math., Oxford Ser. **10**, 252–260 (1939); these Rev. **1**, 140] the author shows that the following function is self-reciprocal in the J_λ-transform:

$$x^{\frac{1}{2}(\mu+\nu-1)}e^{\frac{1}{2}x^2}\sum_{r=0}^{\infty}\frac{(\frac{1}{2}\mu-\frac{1}{2}\nu,r)(1+\mu,r)}{r!(1+\frac{1}{2}\mu+\frac{1}{2}\lambda,r)}x^rW_{k,l}(\tfrac{1}{2}x^2),$$

where $k=-\frac{3}{4}\mu-\frac{1}{4}\nu-\frac{1}{2}$, $l=\frac{1}{4}\mu-\frac{1}{4}\nu+\frac{1}{2}r$, $W_{k,l}$ is Whittaker's function, $\nu-\mu$ is an even positive integer, $\mu\geqq-1$, $\nu>-1$, and (α,r) denotes, as usual, $\alpha(\alpha+1)\cdots(\alpha+r-1)$. Simpler functions are obtained by putting successively $\mu=\lambda$ or 0, then $\nu=\pm\lambda-2$; e.g.,

$$x^{\lambda-\frac{1}{2}}e^{\frac{1}{2}x^2}\sum_{r=0}^{\infty}x^rW_{-\lambda-\frac{1}{2}r,\frac{1}{2}+\frac{1}{2}r}(\tfrac{1}{2}x^2)$$

is self-reciprocal. *R. P. Boas, Jr.* (Chapel Hill, N. C.).

Mohan, Brij. Formulae connecting self-reciprocal functions. Indian J. Phys. **15**, 337–341 (1941). [MF 6507]

The author has given [Proc. London Math. Soc. (2) **34**, 231–240 (1932); Proc. Edinburgh Math. Soc. (2) **4**, 53–56 (1934)] conditions under which the integral transformation $g(x)=\int_0^\infty P(xy)f(y)dy$ transforms the class R_μ of functions self-reciprocal in the Hankel transform of order μ into the class R_ν. He gives several examples of kernels $P(x)$ satisfying the conditions: for example, $x^{\frac{1}{2}-\nu}J_{\nu-\frac{1}{2}}(\frac{1}{2}x)J_{\nu-\frac{1}{2}}(\frac{1}{2}x)$ takes $R_{2\nu-\frac{1}{2}}$ into R_ν if $\nu>\frac{3}{4}$. By means of these kernels, specific self-reciprocal functions are obtained: for example,

$$x^{\nu-\frac{1}{2}}e^{-\frac{1}{2}x^2}{}_1F_1(\tfrac{1}{2};\nu+\tfrac{1}{2};\tfrac{1}{2}x^2)$$

is R_ν for $\nu>\frac{3}{4}$. *R. P. Boas, Jr.* (Chapel Hill, N. C.).

Shanker, Hari. On functions which are Hankel-transforms of each other. J. Indian Math. Soc. (N.S.) **5**, 109–112 (1941). [MF 6127]

If the function $\Phi(m,n;x)$ is defined by

$$\Phi(m,n;x)=m!2^me^{-x^2/2}x^{2n+\nu+1/2}L_m^{n+\nu}(x^2/2),$$

where m and n are positive integers, the author proves that its Hankel transform of order ν is the function $\Phi(n,m;x)$. Thus for $m=n$ the function is self-reciprocal. Other special cases are discussed briefly. *M. C. Gray.*

* **Achyeser, N. Lectures on the Theory of Approximation.** Kharkoff, 1940. 136 pp. (Russian)

This book offers an elegant exposition of fundamental facts of the theory of approximation to a given function by linear aggregates of functions of a given family, and, despite being fairly short, contains many results which are absent in other books on the subject. Chapter 1 (Questions of approximation in a linear normed space) establishes existence theorems and uniqueness theorems in case the space is strictly normed. Particular attention is concentrated on Hilbert spaces, but many other spaces, such as C, L_p, etc., are also treated. Among other things we find here also a theorem of Müntz giving necessary and sufficient conditions for the closure of the family $\{t^{p_k}\}$, $p_k>-\frac{1}{2}$, $\lim_k p_k=\infty$. The second chapter (Range of ideas of P. Tschebyscheff) considers as a central problem the problem of approximation to a given function $f(x)$ by a function of the type $Q(x)=S(x)P_n(x)/P_m(x)$, where $S(x)$ is a given continuous function different from zero, and P_n, P_m are polynomials of degrees n and m, respectively. The author considers various interesting particular cases. At the end of the chapter he considers a general problem of Haar (necessary and sufficient conditions on $\{f_k(P)\}$ in order that the polynomial of best approximation $\sum_1^n x_if_i(P)$ to any continuous function $f(P)$ should be unique). Also, some problems of approximation theory in the space L_1 (which is not strictly normed) are discussed. The content of the last chapter 3 (On best approximation by trigonometrical sums to some classes of periodic functions) is clear from its title. Together with classical results of S. Bernstein, de la Vallée Poussin, Jackson, Fejér, the author also discusses some refinements of the theory, due mainly to himself and Favard.

J. D. Tamarkin (Providence, R. I.).

Miranda, C. Alcune generalizzazioni delle serie di funzion ortogonali e loro applicazioni. Univ. e Politecnico Torino Rend. Sem. Mat. Fis. **7**, 3–17 (1941). [MF 6277]

A set $\{\varphi_i(x)\}$ of functions continuous in $(0,1)$ is called pseudo-orthonormal if

$$\int_0^1 \varphi_i(x)\varphi_k(x)dx+L[\varphi_i]\cdot L[\varphi_k]=\delta_k^i,$$

where L is a linear operator from continuous functions to an r-dimensional Euclidean vector space, and the dot denotes the scalar product. The author recalls how such systems arise from physical problems (discussing particularly the problem of a weighted vibrating string and the solution of weighted integral equations); he gives numerous references. In the special case where the operator L is defined by $L[f]=(p_1f(a_1),\cdots,p_rf(a_r))$ (p_i real, $0\leqq a_i\leqq1$), the author discusses convergence in mean of the series of pseudo-orthonormal functions corresponding to a given function. He seems not to have noticed that the whole discussion for this class of operators L can be brought back to the classical case by regarding the functions $\varphi_i(x)$ as orthonormal with respect to the weight function $\alpha(x)=x+\beta(x)$, where $\beta(x)$ has the jump p_i^2 at the point a_i; here orthonormality is defined by

$$\int_0^1 \varphi_i(x)\varphi_k(x)d\alpha(x)=\delta_k^i.$$

R. P. Boas, Jr. (Chapel Hill, N. C.).

Churchill, R. V. Expansions in series of non-orthogonal functions. Bull. Amer. Math. Soc. **48**, 143–149 (1942). [MF 6194]

If the derivative of $f(x)$ exists, is bounded and L-integrable in $0 \leqq x \leqq 1$, then the author shows that $f(x)$ can be represented in the form

$$f(x) = \sum_0^\infty A_i y_i(x), \qquad 0 \leqq x \leqq 1.$$

The $y_i(x)$ are the characteristic functions of the differential system

$$y'' + (\lambda + q(x))y = 0,$$
$$a_i y(i) + y'(i) + b_i y''(i) = 0, \qquad i = 0, 1,$$

with $a_0, a_1, b_0 \leqq 0$, $b_1 \geqq 0$ constants. The constants A_i depend on $f(x)$ and the $y_i(x)$. *F. G. Dressel* (Durham, N. C.).

Titchmarsh, E. C. On expansions in eigenfunctions. VI. Quart. J. Math., Oxford Ser. **12**, 154–166 (1941). [MF 5981]

In a previous paper [Quart. J. Math., Oxford Ser. **12**, 33–50, 89–107 (1941); these Rev. **3**, 121] the author considered the operator $d^2/dx^2 - q(x)$ and developed a direct method of treating the related expansion problem for an interval one end of which is at infinity or at a singular point of $q(x)$. In the present paper it is shown that the method can be extended to the case where the interval extends in both directions to infinity or to a singularity of $q(x)$. Analogous results were obtained by H. Weyl [Nachr. Ges. Wiss. Göttingen **1910**, 442–467] by means of the theory of singular integral equations. *J. D. Tamarkin.*

Polynomials, Polynomial Approximations

Duffin, R. J. and Schaeffer, A. C. A refinement of an inequality of the brothers Markoff. Trans. Amer. Math. Soc. **50**, 517–528 (1941). [MF 5721]

A theorem of W. Markoff states that, if $f(x)$ is a polynomial of degree n such that $|f(x)| \leqq 1$ for $-1 \leqq x \leqq 1$, then in the same range we have $|f^{(p)}(x)| \leqq T_n^{(p)}(1)$ $(p = 1, 2, \cdots, n)$, where $T_n(x) = \cos(n \arccos x)$. The authors prove the following refinement: the conclusions of Markoff's theorem are still valid provided we assume that $|f(x)| \leqq 1$ only for the following $n+1$ values $x = \cos(k\pi/n)$ $(k = 0, 1, \cdots, n)$. The proof is based on a number of results typical of which is the following theorem: Let $g(z)$ be a polynomial of degree n with n distinct real roots less than b and such that (1) $|g(x+iy)| \leqq |g(b+iy)|$ if $a \leqq x \leqq b$, $-\infty < y < \infty$. If $f(z)$ is a real polynomial of degree n such that (2) $|f'(x)| \leqq |g'(x)|$ whenever $g(x) = 0$, then (3) $|f^{(p)}(x+iy)| \leqq |g^{(p)}(b+iy)|$ $(p = 1, 2, \cdots, n)$ if $a \leqq x \leqq b$, $-\infty < y < \infty$. The derivation of such strong conclusions concerning $f(x)$ as the inequalities (3) from such weak assumptions as (2) is characteristic of the contents of the paper. The connection between the two theorems just stated is suggested by the fact that $g(z) = T_n(z)$ is shown to enjoy the property (1) for $a = -1$, $b = +1$.

[It should be added that a similar refinement concerning Tchebycheff's theorem (If $|f(x)| \leqq M$ in $-1 \leqq x \leqq 1$, where $f(x) = 2^{n-1}x^n + \cdots$, then $M \geqq 1$, while $M = 1$ implies $f(x) \equiv T_n(x)$) seems to be well known. Shohat pointed out to the reviewer his paper [Math. Z. **29**, 684–695 (1929)], which contains similar refinements for the case of polynomials deviating least from zero in $(-1, 1)$ among polynomials whose coefficients satisfy a fixed linear relation.] *I. J. Schoenberg* (Philadelphia, Pa.).

Ward, A. J. Some inequalities connected with convergent sequences. J. London Math. Soc. **16**, 13–23 (1941). [MF 5899]

If $\{a_n\}$ is a given sequence of complex numbers with limit zero, let $P(z)$ be a polynomial of degree n with highest coefficient 1 and with zeros exclusively among the points a_n. Let $M_n = \sup |P_n(z)|^{1/n}$ for $z \varepsilon \{a_n\}$, and let ρ_n be the lower bound of M_n for all polynomials of this class. Inequalities of R. Wilson are obtained which relate the growth of ρ_n^{-1} with the rate of growth of $|a_n|^{-1}$. It is shown that under a separation condition on the given sequence, namely, $|a_i - a_j| \geqq |a_i|^{1+\delta}$, more precise results can be obtained. For example, if $\log |a_n| \sim A \log n$, then $\log \rho_n \sim A \log n$. *A. C. Schaeffer* (Stanford University, Calif.).

Sebastião e Silva, J. Sur une méthode d'approximation semblable à celle de Gräffe. Portugaliae Math. **2**, 271–279 (1941). [MF 6371]

Let $F(x) = x^n + \cdots$ be a polynomial of degree n with distinct roots α_i such that $|\alpha_i| > |\alpha_j|$ $(i = 1, \cdots, r; j = r+1, \cdots, n)$. Let $f_1^p(x)$, of degree not greater than $n-1$, be the remainder of the division of x^p by $F(x)$. Let $f_2^p(x)$, of degree not greater than $n-2$, be the remainder of $F(x)$ by $f_1^p(x)$, and continue this process until we get $f_r^p(x) = c_p x^{n-r} + \cdots$ of degree not greater than $n-r$. It is shown that as $p \to \infty$ the polynomial $c_p^{-1} f_r^p(x) \to (x - \alpha_{r+1}) \cdots (x - \alpha_n)$. This result leads to a method of approximate solution of equations having some features in common with Graeffe's method, the effectiveness of the latter not being one of them. No numerical example is given. *I. J. Schoenberg.*

Thomas, Joseph Miller. Sturm's theorem for multiple roots. Nat. Math. Mag. **15**, 391–394 (1941). [MF 5690]

Sturm's theorem is usually stated for polynomials with only simple roots. If the algorithm is applied to a polynomial with multiple roots and its last term (the G.C.D. of f and f') is divided out of the sequence, then the loss in variations for this new sequence will count the roots without their multiplicities. The author gives a simple rule for setting up an extended Sturm sequence whose loss in variations will under all circumstances count the roots, including their multiplicities, in half-open intervals $a < x \leqq b$. Moreover, subsections of this sequence indicate the number of roots of given multiplicity. *I. J. Schoenberg.*

Weisner, Louis. Polynomials whose roots lie in a sector. Amer. J. Math. **64**, 55–60 (1942). [MF 5988]

From $A(x) = a_0 + \cdots + a_m x^m$, $B(x) = b_0 + \cdots + b_m x^m$ form $f(x) = a_0 b_0 + \cdots + a_s b_s x^s$, $g(x) = a_0 b_0 + 1! a_1 b_1 x + \cdots + s! a_s b_s x^s$, $s = \min(m, n)$. If the roots of $A(x) = 0$ are real and the roots of $B(x) = 0$ are real and of like sign, then [Malo, 1895] the roots of $f(x) = 0$ and [Schur, 1914] of $g(x) = 0$ are real. The author proves the generalization: if the roots of $A(x) = 0$ are real while those of $B(x) = 0$ lie in a sector S of the x-plane starting at the origin, then the roots of $f(x) = 0$ and of $g(x) = 0$ lie in the double sector $\pm S$. The concluding paragraph is due to a suggestion by Pólya, and extends these results to the following case: $A(x) = a_0 + \binom{n}{1} a_1 x + \cdots + \binom{n}{n} a_n x^n$, $B(x) = b_0 + \binom{n}{1} b_1 x + \cdots + \binom{n}{n} b_n x^n$, $h(x) = a_0 b_0 + \binom{n}{1} a_1 b_1 x + \cdots + \binom{n}{n} a_n b_n x^n$, where the same assumptions concerning the roots of $A(x) = 0$, $B(x) = 0$

are made as above. Special precautions have to be taken for zero and infinite roots. The connection with many known theorems is pointed out in footnotes.

A. J. Kempner (Boulder, Colo.).

Erdős, P. On the uniform distribution of the roots of certain polynomials. Ann. of Math. (2) **43**, 59–64 (1942). [MF 6060]

Continuation of investigations and improvements of results contained in a previously published paper by the author and Turán [Ann. of Math. (2) **41**, 162–173 (1940); these Rev. **1**, 217]. *M. Kac* (Ithaca, N. Y.).

Gavriloff, L. Sur la K-prolongation des polynômes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 235–236 (1941). [MF 5857]

Let $f(z)=1+a_1z+\cdots+a_nz^n$ be a given polynomial. It is shown that new terms may so be added that the polynomial $f_1(z)=f(z)+a_{n+1}z^{n+1}+\cdots+a_mz^m$ will have all its zeros on the unit circle $K: |z|=1$. This extension is also possible if the zeros of $f_1(z)$ are to avoid an arbitrarily given set of measure zero on K; if $f(z)$ is real also its extension $f_1(z)$ can be made real. *I. J. Schoenberg.*

Tschebotaröw, N. G. Über die Fortsetzbarkeit von Polynomen auf geschlossene Kurven. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 3–6 (1941). [MF 5843]

The author proves the following theorem: let C be a closed star-shaped Jordan curve containing the origin in its interior. Any polynomial $f(z)=1+a_1z+\cdots+a_nz^n$ being given, it is possible to find a polynomial $\phi(z)=b_1z^{n+1}+\cdots+b_mz^{n+m}$, such that $f(z)+\phi(z)$ has all its zeros on C. This statement is equivalent to the following one [see Tschebotaröw, Bull. [Izvestia] Soc. Phys.-Math. Kazan (3) **8**, 103–108, 109–123 (1936–1937)]: the quantities s_ν $(\nu=1, 2, \cdots, n)$ being defined by $s_1+a_1=0$, $s_2+a_1s_1+2a_2=0$, $\cdots$, $s_n+a_1s_{n-1}+\cdots+na_n=0$, there exist quantities α_r $(r=1, \cdots, p)$, all situated on the curve C' obtained from C by the transformation $U=1/z$, and such that $s_k=\sum_{r=1}^{p}\alpha_r^k$ $(k=1, 2, \cdots, r)$. *S. Mandelbrojt.*

Sutton, Charles S. A study of certain polynomials. Bull. Citadel **5**, 21–32 (1941). [MF 6283]

A study is made of the polynomial $P_n(x)$ defined by the relation

$$(1)\quad F_n(x)=1+2^nx+3^nx^2+\cdots =P_n(x)/(1-x)^{n+1},\quad n=0, 1, 2, \cdots,$$

the coefficients and zeros of $P_n(x)$, contour-integral representation, asymptotic behavior $(n\to\infty)$, relation to Bernoulli polynomials, etc. The main point is the recurrence relation $P_{n+1}(x)=(1+nx)P_n(x)+x(1-x)P_n'(x)$, which holds, by (1), for $|x|<1$, hence for any x [since $P_n(x)$ is a polynomial, and not "by analytic continuation." Here and in other places notations and reasoning seem unnecessarily complicated. Thus, on page 23 we read $P_n(x)=\sum_{m=0}^{n}a_{nm}x^m$, while on page 21 it is stated that the degree of $P_n(x)$ does not exceed n. It seems also advantageous to use the relations: $F_1(x)=F_0'(x)$, $F_2(x)=F_0'(x)+xF_0''(x)$, $F_3(x)=F_0'(x)+3xF_0''(x)+x^2F_0'''(x)$, where $F_0(x)=1/(1-x)$. Reviewer]. *J. A. Shohat* (Philadelphia, Pa.).

Dickinson, D. R. On Tchebycheff polynomials. II. Quart. J. Math., Oxford Ser. **12**, 184–192 (1941). [MF 5985]

The system $\phi_0(x), \cdots, \phi_n(x)$ of real functions continuous in a closed interval $[a, b]$ is called a T-system if no linear combination $P(x)=\sum A_i\phi_i(x)$, where not all A_i vanish, has more than n distinct zeros in $[a, b]$. The point $\bar{x}$ is called a simple zero of $P(x)$ if $P(x)$ changes sign as x passes through $\bar{x}$. It is called a weak double zero if $P(x)$ preserves the sign, and a strong double zero if, in addition, $P(x)/Q(x)\to 0$ as $x\to\bar{x}$ for any polynomial $Q(x)$ having a simple zero at $\bar{x}$, and at least $(n-2)$ other zeros. Finally, $\bar{x}$ is a weak triple zero if $\bar{x}$ is interior to (a, b), if $P(x)$ changes sign, and $P(x)/Q(x)\to 0$ as $x\to\bar{x}$ for any polynomial $Q(x)$ having a simple zero at $\bar{x}$, and at least $(n-2)$ other zeros; $\bar{x}$ is a strong triple zero if, in addition, the previous relation is true for any $Q(x)$ having $\bar{x}$ as a double zero, and at least $(n-3)$ other zeros. Extending results of his previous paper [Quart. J. Math., Oxford Ser. **10**, 277–282 (1939); these Rev. **1**, 143], where double zeros were discussed, the author proves that, if s, d, t are, respectively, the number of simple, double, triple zeros of P, then $s+2d+3t\leqq n$. Furthermore, if $s+2d+3t=n$, and we are given $s+d+t$ distinct points $x_1, \cdots, x_s; y_1, \cdots, y_d; z_1, \cdots, z_t, a\leqq x_i\leqq b, a<y_j<b, a<z_k<b$, then it is possible to construct a polynomial $P(x)$ having simple zeros at x_i, weak double zeros at y_j and weak triple zeros at z_k. *J. D. Tamarkin* (Providence, R. I.).

Banerjee, D. P. On the expansion of a function in a series of Sonine's polynomials. Bull. Calcutta Math. Soc. **32**, 61–64 (1940). [MF 6157]

The author claims that as yet no method has been given for the expansion of a function in a series of Sonine polynomials ($T_m^n(z)$ of degree n and order m) and proceeds to give such a method applicable to entire functions of exponential type. The first expansion of the form $\sum c_nT_m^n(z)$ as well as the method is in the literature for the equivalent Laguerre polynomials $L_n^{(m)}(z)$ [cf., for $m=0$, F. Tricomi, Atti Accad. Naz. Lincei. Rend. (6) **21**, 232–239 (1935)]. The author's proof is partly erroneous and does not bring out the essential restriction that the Laplace transform of the function must be analytic for $\Re(w)\geqq\frac{1}{2}$. The second expansion $\sum c_kz^kT_k^n(z)$ appears to be new. There are disturbing misprints at several important points.

E. Hille (New Haven, Conn.).

Steffensen, J. F. The poweroid, an extension of the mathematical notion of power. Acta Math. **73**, 333–366 (1941). [MF 6099]

The power function x^r satisfies the condition $Dx^r=rx^{r-1}$, where D denotes the operation of differentiation with respect to x. A "poweroid," denoted by $x^{\overline{r}|}$, is defined as a polynomial of degree r such that $x^{\overline{0}|}=1$, $0^{\overline{r}|}=0$ $(r>0)$ and $\theta x^{\overline{r}|}=rx^{\overline{r-1}|}$ $(r>0)$, where θ is an operator of the form

$$\theta=\varphi(D)=\sum_{\nu=1}^{\infty}k_\nu D^\nu,\qquad k_1\neq 0,$$

the expansion being convergent if the symbol of differentiation D is replaced by a sufficiently small number. The author establishes the existence and uniqueness of the poweroids $x^{\overline{r}|}$ $(r=0, 1, 2, \cdots)$ corresponding to any given operator θ. He obtains an expansion

$$f(x)=\sum_{\nu=0}^{\infty}\theta^\nu f(0)x^{\overline{\nu}|}/\nu!,$$

always valid if $f(x)$ is a polynomial, and having an obvious analogy with Maclaurin's series. A variety of expressions are derived for $x^{\overline{r}|}$ by which the poweroid can be computed

when θ is given. A recurrence formula for passing from $x^{\overline{r}}$ to $x^{\overline{r+1}}$ is also given, as well as relations between the poweroids belonging to two different θ's. The operation which is the inverse of θ may be called "θ-integration," and, like ordinary integration, is not unique owing to the presence of an arbitrary additive constant. If we set $D=t$, $\zeta(0)=0$ and $\zeta=\varphi(t)$ then

$$e^{xt}=\sum_{\nu=0}^{\infty} x^{\overline{\nu}}\zeta^{\nu}/\nu!,$$

so that e^{xt}, if expanded in powers of ζ, may be regarded as the generating function of the poweroids. The author shows how the factorials $x^{(\nu)}$ and $x^{(-\nu)}$ are obtained from the appropriate θ-operators, and in addition poweroids are found which are related to the polynomials of Hermite, Laguerre, Bernoulli and Euler. Finally consideration is given to the meaning of the operation $\theta f(x)$ when $f(x)$ is not a polynomial. *W. E. Milne* (Corvallis, Ore.).

*Wintner, Aurel. **Spherical equidistributions and a statistics of polynomials which occur in the theory of perturbations.** Astronomical Papers dedicated to Elis Strömgren, pp. 287–297. Einar Munksgaard, Copenhagen, 1940.

Strömgren has observed that the Gegenbauer polynomial $C_n^{\nu}(\cos\theta)$ with positive ν is an even trigonometric polynomial with positive coefficients. From this it follows that a suitable multiple of the Gegenbauer polynomial may be regarded as a Fourier-Stieltjes transform in which the distribution function is a step-function whose jumps are proportional to the coefficients of the trigonometric polynomial. For $n\to\infty$, $C_n^{\nu}(\cos u/n)$ tends to a multiple of a Bessel function of the first kind, and the aforementioned step-function tends to the (known) absolutely continuous distribution function connected with the multiple of the Bessel function. This gives some information about the asymptotic behavior of the coefficients of $C_n^{\nu}(\cos\theta)$. The limiting (absolutely continuous) distribution function has a statistical meaning as the one-dimensional distribution which represents the projection on a diameter of the equidistribution on the boundary of the $(2\nu+1)$-dimensional unit sphere. The distribution function which represents the projection on a "diameter" of the equidistribution on Riemann's elliptic space of constant curvature has as Fourier transform the multiple of a modified Bessel function of the third kind. Both distribution functions also appear in the theory of small samples, the second in connection with "Student's ratio." *A. Erdélyi* (Edinburgh).

Special Functions

Erdélyi, A. Note on Heine's integral representation of associated Legendre functions. Philos. Mag. (7) **32**, 351–352 (1941). [MF 5916]

By a slight modification of the integral

$$\int_0^{2\pi}(z+(z^2-1)^{\frac{1}{2}}\cos\phi)^n\cos m\phi\,d\phi=\text{const. } P_n^m(z)$$

its validity can be extended to unrestricted values of m and $\Re n>-1$. The author writes $e^{-im\phi}$ instead of $\cos m\phi$ and integrates along a suitable eight-shaped contour.

G. Szegö (Stanford University, Calif.).

Shabde, N. G. On some results involving Legendre functions. Bull. Calcutta Math. Soc. **32**, 121–124 (1940). [MF 6162]

Two integrals involving a power of $1-x^2$ and a product of associated Legendre functions of the first or second kind are evaluated for the range $-1, +1$. A similar integral involving a single associated Legendre function and powers of $1+x$ and $1-x$ is expressed as a series involving the polynomial $F_n^m(z)$ and the result is extended to an integral involving a product of two associated Legendre functions. Other integrals which are evaluated involve Bessel functions and a logarithmic function. *H. Bateman.*

Mitra, S. C. On a certain new connection between Legendre functions and Whittaker's M-functions. Bull. Calcutta Math. Soc. **31**, 161–162 (1939). [MF 6150]

The definite integral

$$\int_0^1 P_n(1-2x^2)(xz)^{-2m}e^{-\frac{1}{2}x^2z^2}M_{k,m}(x^2z^2)\,dx$$

(n zero or a positive integer, k and m unrestricted) is evaluated in the form of an infinite series which reduces to a Whittaker M-function for particular values of m and k.

M. C. Gray (New York, N. Y.).

Tsvetkoff, G. E. On roots of Whittaker's functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 10–12 (1941). [MF 5845]

The author states, without proof, various theorems on the number and location of the roots of the functions $M_{k,m}(z)$ and $W_{k,m}(z)$ for real values of m and k. For example, introducing the function $E(q)=0$ if $q\geqq 0$, $E(q)=[q]$ if $q<0$, and the integer $N_{\pm}=-E(\frac{1}{2}+m\mp k)+E(2m+1)$, a formula for the number of positive roots $X_1(m,k)$ and of negative roots $X_2(m,k)$ of $M_{k,m}(z)$ is $X_{1,2}(m,k)=N_{\pm}$, if $N_{\pm}\geqq 0$; $=1$ if $N_{\pm}$ is an odd negative integer and $\frac{1}{2}+m\mp k$ is not an integer; $=0$ if $N_{\pm}$ is an even negative integer, and if $\frac{1}{2}+m\mp k$ is an integer and $N_{\pm}$ is negative. For the function $W_{k,m}(z)$, with $m>0$, the number of positive roots is $N'=-E(\frac{1}{2}+m-k)$ if $N'>0$ and is zero if $N'\leqq 0$.

M. C. Gray (New York, N. Y.).

Feldheim, Ervin. Trasformata di Hankel di funzioni di Whittaker. Ann. Scuola Norm. Super. Pisa (2) **10**, 103–114 (1941). [MF 6416]

In an earlier paper [Ann. Scuola Norm. Super. Pisa (2) **9**, 225–252 (1940); these Rev. **3**, 112] the author gave a functional equation for hypergeometric functions of several variables. He is now concerned with certain limiting cases which lead to confluent functions. He obtains various expressions for integrals of the form

$$\int_0^{\infty} x^{\lambda-1}e^{-x}J_{\mu}(2(xy)^{\frac{1}{2}})J_{\nu}(2(xz)^{\frac{1}{2}})\,dx,$$

$$\int_0^{\infty} x^{\mu-1}e^{-x(1-z)}J_{\nu}(2(xy)^{\frac{1}{2}})M_{k,m}(2xz)\,dx,$$

where $M_{k,m}(t)$ is the Kummer-Whittaker function. The author specifies the obvious restrictions which have to be imposed on the parameters λ, μ, ν, but he fails to note that the second integral diverges for $z>\frac{1}{2}$ since $M_{k,m}(t)>C(k,m)t^{-k}\exp(\frac{1}{2}t)$ for large $t>0$. As a consequence the "relazioni molto interessante" on pp. 106–107, corresponding to $z=1$, have no foundation in fact. Moreover, a formula

like the author's number (12)

$$\int_0^\infty J_{2m}(2(xy)^{\frac{1}{2}})I_m(x)dx=e^{\pi i(m+\frac{1}{2})}I_m(y)$$

cannot be true for any reasonable interpretation of the divergent integral since the integrand is real when $y>0$ and $m>0$ while the right hand side in general is not.

E. Hille (New Haven, Conn.).

Shastri, N. A. Some results involving Bateman's polynomials. Bull. Calcutta Math. Soc. **32**, 89–94 (1940). [MF 6160]

Some integrals involving Bessel functions, exponential functions and the function $k_{2m}(x)$ are expressed as series involving the function $k_{2n}(z)$ or products of two such functions with different arguments and suffixes. Some expansions are given also for the functions of the parabolic cylinder in series of Whittaker's confluent hypergeometric function. Expressions involving the ber and bei functions are finally expanded in series of the polynomials of Sonine and Laguerre. *H. Bateman* (Pasadena, Calif.).

Schrödinger, Erwin. The factorization of the hypergeometric equation. Proc. Roy. Irish Acad. Sect. A. **47**, 53–54 (1941). [MF 6451]

If in the hypergeometric equation

$$x(1-x)y''+(c-(a+b+1)x)y'-aby=0$$

we put

$$2x=1+\cos t,\quad y=z(2x)^{\frac{1}{2}(1-2c)}(2-2x)^{-\frac{1}{2}(a+b-c+\frac{1}{2})},$$

the differential equation for z may be written in the form

$$0=(D_t+A\ \operatorname{cosec} t+B\cot t)(D_t-A\ \operatorname{cosec} t-B\cot t)z+Cz,$$

where A, B and C can be expressed simply in terms of a, b and c. Four alternatives are offered. *H. Bateman.*

Jackson, F. H. Hypergeometric series and "set" numbers. Quart. J. Math., Oxford Ser. **12**, 201–210 (1941). [MF 6545]

The author develops the idea of a set number (n) which represents the sum of the elements $p_1+p_2+\cdots+p_n$ of an infinite sequence of elements p_m. This is somewhat analogous to the symbol $[n]$ used by Morgan Ward for the product $p_1p_2\cdots p_n$ [Amer. J. Math. **58**, 255–266 (1936)]. The idea of a basic "set" number $[(n)]=(q^{(n)}-1)(q-1)^{-1}$ is introduced and a correspondence is found between the basic "set" series and the basic hypergeometric series of Heine. A theorem of M. J. M. Hill on the truncated hypergeometric series is found also to have a counterpart in the case of the general series involving set numbers.

H. Bateman (Pasadena, Calif.).

Jackson, F. H. Certain q-identities. Quart. J. Math., Oxford Ser. **12**, 167–172 (1941). [MF 5982]

The identities in question give finite expressions for certain finite power series in q, the coefficients alternating in sign and involving a q-analogue of the binomial coefficient and a product of quantities of type $(a)_r$, $(b)_{2n-r}$, where

$$(a)_n=(a-1)(aq-1)\cdots(aq^{n-1}-1).$$

A q-generalization is also given of Dougall's identity in the theory of hypergeometric functions. *H. Bateman.*

Bailey, W. N. A note on certain q-identities. Quart. J. Math., Oxford Ser. **12**, 173–175 (1941). [MF 5983]

The q-identities of F. H. Jackson are regarded as theorems in the theory of basic hypergeometric series.

H. Bateman (Pasadena, Calif.).

Banerjee, D. P. The expansion of an arbitrary function in a series of toroidal functions of the second kind. Amer. J. Math. **64**, 72–80 (1942). [MF 5990]

A function $f(z)$ which is analytic for $|z|>1$ and such that $\lim_{z\to\infty} z^{\frac{1}{2}}f(z)=0$ can be expanded in a convergent series of toroidal functions of the second kind, that is,

$$f(z)=\sum_0^\infty a_nQ_{n+\frac{1}{2}}(z).$$

The author introduces an auxiliary function $\Phi(h)$, defined by

$$f(z)=\int_0^{z-(z^2-1)^{\frac{1}{2}}}\frac{\Phi(h^{\frac{1}{2}})dh}{(1-2zh+h^2)^{\frac{1}{2}}},$$

and determines Φ by the formal expression

$$\Phi(h^{\frac{1}{2}})=\frac{2(1-h^2)}{(1+h^2)^{\frac{1}{2}}}\Phi_1'\left[\left(\frac{2h}{1+h^2}\right)^{\frac{1}{2}}\right],$$

where

$$\Phi_1'(u)=-\frac{2}{\pi}\left(I+u\frac{dI}{du}\right),$$

$$I=\int_0^{\pi/2}\frac{F(u^2\sin^2\psi)}{2u\sin\psi}d\psi,\quad F(u)=u^{-\frac{1}{2}}f(1/u).$$

Then, if $\Phi(h)$ is an odd analytic function of h, the expansion $\Phi(h^{\frac{1}{2}})=\sum a_nh^{n+\frac{1}{2}}$ determines the coefficients a_n. The author obtains similar expansions in series of the associated toroidal functions $Q_{n+\frac{1}{2}}^m(z)$, and also an expansion

$$f(\cos\theta)=\sum b_nQ_{n+\frac{1}{2}}(\cos\theta),\qquad 0<\theta\leqq\pi/2.$$

M. C. Gray (New York, N. Y.).

Mohan, B. Some infinite integrals. J. Indian Math. Soc. (N.S.) **5**, 123–127 (1941). [MF 6131]

The author evaluates some definite integrals containing the function $\varphi_\nu(x)$ defined for $\Re(\nu)>-1$ as

$$\sum_{r=0}^\infty\frac{(-1)^r\Gamma(\frac{1}{2}+r)x^{\nu+2r}}{r!2^{\nu+2r}\Gamma(1+\nu+r)}.$$

For example,

$$\int_0^\infty x^{p-1}\varphi_\nu(ax)dx=\frac{2^{p-1}}{a^p}\frac{\Gamma(\frac{1}{2}p+\frac{1}{2}\nu)\Gamma(\frac{1}{2}-\frac{1}{2}p-\frac{1}{2}\nu)}{\Gamma(1-\frac{1}{2}p+\frac{1}{2}\nu)}$$

for $\Re(a)>0$, $0<\Re(\nu+p)<1$, $\Re(\nu)>-1$. The integral

$$\int_0^\infty x^{-p}\varphi_{p+\nu+1}(ax)J_\nu(bx)dx$$

is expressed in terms of Whittaker functions. The integrals

$$\int_0^\infty x\varphi_m(ax)\varphi_m(bx)dx,\quad \int_0^\infty x^{\frac{1}{2}}\varphi_\lambda(ax)\sin\tfrac{1}{4}bx^2dx,$$

$$\int_0^\infty x^{\frac{1}{2}}\varphi_\lambda(ax)\cos\tfrac{1}{4}bx^2dx$$

are also evaluated. Numerous special choices of the parameters are discussed, some of them leading to well-known formulas. *R. P. Boas, Jr.* (Chapel Hill, N. C.).

Erdélyi, A. On algebraic Lamé functions. Philos. Mag. (7) **32**, 348–350 (1941). [MF 5915]

The author is concerned with the study of the solutions of Lamé's equation

$$\frac{d^2y}{du^2}+(h-n(n+1)k^2\operatorname{sn}^2u)y=0$$

which are nonrational algebraic functions of sn u, cn u, dn u. Such solutions exist if n is the half of an odd integer and h belongs to a set of $n-\frac{1}{2}$ characteristic values. The note follows the methods of a previous note on polynomial solutions of Lamé's equation [Philos. Mag. (7) 31, 123–130 (1941); these Rev. 2, 285]. *G. Szegö.*

Erdélyi, A. On certain expansions of the solutions of Mathieu's differential equation. Proc. Cambridge Philos. Soc. 38, 28–33 (1942). [MF 6032]

Writing Mathieu's equation in the form

$$u''+(a-2\theta\cos 2x)u=0,$$

and imposing no restrictions on the parameters a and θ, the author obtains the series expansion

$$u(x)=\sum_{-\infty}^{\infty} c_n\Phi_{2n-\mu}(x),$$

where

$$\Phi_\nu(x)=\left[e^{ix}\frac{\cos(x-h)}{\cos(x+h)}\right]^{\nu/2}J_\nu(2[\theta\cos(x-h)\cos(x+h)]^{\frac{1}{2}}),$$

and the c's satisfy the recurrence relation

$$\theta c_{n-1}+((2n-i\mu)^2-a)c_n+\theta c_{n+1}=0,\quad n=0,\pm1,\pm2,\cdots.$$

The parameter h is arbitrary and the series converges if $|\cos(x-h)|>1$. Thus the domain of convergence in the x-plane consists of two parts, one entirely above and the other entirely below the line $x=h$. Since these parts have no common point the series represents two distinct solutions of the original differential equation. For special values of h, $h=0$, $h=\pi/2$, $h\to i\infty$, the expansion reduces to generalized forms of known expansions for Mathieu functions (for which the characteristic exponent μ has the value 0 or i). Series involving Bessel functions of the third kind in place of those of the first kind are also obtained, the coefficients c_n being unchanged, and the convergence condition now being $|\cos(x\pm h)|>1$. *M. C. Gray.*

Erdélyi, A. On some generalisations of Laguerre polynomials. Proc. Edinburgh Math. Soc. (2) 6, 193–221 (1941). [MF 5862]

The point of departure is work done by H. Kober and the author on Laguerre polynomials and the cut Hankel transform [Proc. Edinburgh Math. Soc. (2) 6, 135–146 (1940); Quart. J. Math., Oxford Ser. 11, 212–221 (1940); see these Rev. 2, 43, 192]. Let

$$\mathfrak{H}_{\alpha,m}[f]=\text{l.i.m.}_{\omega\to\infty}\int_0^\omega J_{\alpha,m}[2(2xy)^{\frac{1}{2}}]f(y)dy,$$

where $f(y)\in L_2(0,\infty)$, $J_{\alpha,m}(z)$ is the remainder after m terms in $J_\alpha(z)$, $m=0$ if $\Re(\alpha)>-1$, $-1<\Re(\alpha)+2m<1$ otherwise. If $\psi_n^{(\alpha)}(x)=e^{-x}x^{\frac{1}{2}\alpha}L_n^{(\alpha)}(2x)$, then $\psi_n^{(\alpha)}(x)=(-1)^n\mathfrak{H}_\alpha[\psi_n^{(\alpha)}(x)]$, where $\mathfrak{H}_\alpha=\mathfrak{H}_{\alpha,0}$. The author and H. Kober have shown that, if $G(x)=\mathfrak{H}_{\alpha+2m}[F(x)]$, then it is possible in two different ways to define transforms $g(x)$ and $f(x)$ such that $g(x)=\mathfrak{H}_{\alpha,m}[f(x)]$. One choice is $g(x)=x^{-\frac{1}{2}\alpha}D^m[x^{\frac{1}{2}\alpha+m}G(x)]$, with the same relation between $f(x)$ and $F(x)$; the other choice involves m-fold integrals instead. Substituting here $\psi_n^{(\alpha+2m)}(x)$ for $G(x)$, the author obtains the corresponding functions $g(x)$ denoted by $\chi_n^{(\alpha,m)}(x)$ and $\phi_n^{(\alpha,m)}(x)$, respectively. These functions form a biorthogonal set and are the desired generalization of the Laguerre functions. They are investigated at length, but for the details we have to refer to the original paper. Sections 12 through 15 contain some remarks on Hankel and "general" transforms which are highly suggestive. They concern the rôle of the generating function, its relation to a supposedly known self-reciprocal function $\phi(x)$ of the kernel $K(xy)$, its use in proving orthogonality and closure, and the rôle of the associated transformation

$$T_s[f]=s^{\frac{1}{2}}\int_0^\infty \phi(sx)f(x)dx,$$

introduced by F. Tricomi for the Hankel transform. *E. Hille* (New Haven, Conn..)

Differential Equations

Conte, Luigi. Sull'integrazione dell'equazione differenziale $y'=Ay^3+By^2+Cy+D$. Boll. Un. Mat. Ital. (2) 3, 384–391 (1941). [MF 5707]

A. Chiellini [Pont. Acad. Sci. Comment. 4, 385–411 (1940); cf. ces Rev. 2, 288] a étudié la condition afin que l'équation différentielle considérée puisse se réduire aux quadratures. L'auteur discute certain nombre de cas dans lesquels la réduction s'effectue par des calculs relativement simples, moyennant une substitution de la forme

$$z=y+B/3A\quad\text{ou}\quad z=y+(B\pm(B^2-3AC)^{\frac{1}{2}})/3A.$$

Dans un paragraphe final on obtient, par élimination des coefficients de l'équation, un groupe de relations comparables avec la propriété du birapport constant pour l'équation de Riccati. *B. Levi* (Rosario).

Nachbin, Leopoldo. On the interchangeability of passages to the limit and the integration of differential equations. Anais Acad. Brasil. Sci. 13, 327–335 (1941). (Portuguese) [MF 6589]

Let $y=f_p(x)$ ($p=1, 2, \cdots$) be the solution of $dy/dx=g_p(x, y)$ with $y_0=f_p(x_0)$ and $\lim_{p\to\infty} g_p(x, y)=g(x, y)$. The author gives a sufficient condition for the uniform convergence of $\lim_{p\to\infty} f_p(x)=f(x)$, where $y=f(x)$ is the solution of $dy/dx=g(x, y)$ with $y_0=f(x_0)$. *I. Opatowski.*

Lemke, H. Über die Gleichung der Gaskugeln und andere Differentialgleichungen von ähnlicher Form. J. Reine Angew. Math. 183, 197–231 (1941). [MF 5724]

The author discusses the analytic character of the equation $\eta d\eta/d\xi=\eta+F(\xi)$, where

$$F(\xi)=c(\xi-\alpha_1)(\xi-\alpha_2)\cdots(\xi-\alpha_n),$$

and shows how to find power series solutions in the neighborhood of the critical points $\xi=\alpha_i$. The equation considered includes Emden's equation for a gas sphere as a special case. *B. Friedman* (Chicago, Ill.).

Butlewski, Zygmunt. Sur les intégrales oscillantes et bornées d'une équation différentielle du second ordre. Ann. Scuola Norm. Super. Pisa (2) 9, 187–200 (1940). [MF 5461]

On étudie l'allure des intégrales réelles de l'équation différentielle

$$\frac{d}{dt}\left[\theta(t)\frac{dx}{dt}\right]+f(t, x)=0$$

pour les grandes valeurs de t. Les théorèmes I et II sont des généralisations de propriétés connues en certains cas particuliers [Kneser, Picard]; supposant $\theta(t)$ positive, les

solutions ne peuvent être oscillantes si $f(t, x)$ est de signe opposé à x pour les grandes valeurs de t, tandis qu'elles sont toujours oscillantes si, étant $\lim \theta(t)^{-1}>0$, $\lim f(t, x)$ est >0 ou <0 selons que x tend à un nombre positif ou négatif (on voit facilement que ces limites pourraient encore être substituées convenablement par des limites inférieures et supérieures). Dans la suite, sous des conditions un peu plus restrictives pour la fonction f, on étudie les relations entre les zéros des solutions oscillantes et les zéros de leurs dérivées; on considère en particulier les cas de $f(t, x)$ polynome en x à termes tous de degré impair et de $f(t, x)$ de la forme $A(t)f(x)$. *B. Levi* (Rosario).

Malmquist, J. Sur les fonctions à un nombre fini de branches satisfaisant à une équation différentielle du premier ordre. Acta Math. **74**, 175–196 (1941). [MF 5873]

Differential equations of the form

$$F_0(y, x)y'^m+F_1(y, x)y'^{m-1}+\cdots+F_m(y, x)=0,$$

where $F_0, \cdots, F_m$ are integral rational functions of y in which the coefficients are algebraic functions of x, are considered. The principal theorem of the paper states that, if certain assumptions regarding the critical points are met, then the above equation can be transformed either into an equation of Riccati type

$$z'=a(x)z^2+b(x)z+c(x)$$

by a transformation

$$y^n+R_1(z, x)y^{n-1}+\cdots+R_n(z, x)=0$$

or else into an elliptic differential equation

$$z'^2=a(x)(4z^3-g_2z-g_3)$$

by a transformation

$$y^n+R_1(z, x)y^{n-1}+\cdots+R_n(z, x)+z'[\bar{R}_1(z, x)y^{n-1}+\cdots+\bar{R}_n(z, x)]=0.$$

A type of converse theorem is established. *W. M. Whyburn* (Los Angeles, Calif.).

Malmquist, J. Sur l'étude analytique des solutions d'un système d'équations différentielles dans le voisinage d'un point singulier d'indétermination. III. Acta Math. **74**, 109–128 (1941). [MF 5874]

The present paper is a continuation of the author's two earlier papers of the same title [Acta Math. **73**, 87–129 (1940); **74**, 1–64 (1941); these Rev. **2**, 289]. Using the notation and terminology of the former papers, this paper treats cases where the roots $s_1, \cdots, s_n$ of the characteristic equation are different from zero and satisfy a relation

$$s_i=k_1s_1+\cdots+k_ns_n, \quad k_i \text{ an integer or zero}; \ \sum_1^n k_i\geqq 2.$$

The nth order differential system is first reduced to a normal form which exhibits the first degree terms in the dependent variables. With respect to certain carefully defined sectors, substitutions are developed which change the differential system into new normal forms, demonstrate the existence of solutions and exhibit asymptotic expressions for these solutions. *W. M. Whyburn* (Los Angeles, Calif.).

Cinquini, Silvio. Sopra il problema di Nicoletti per i sistemi di equazioni differenziali ordinarie. Ann. Scuola Norm. Super. Pisa (2) **10**, 127–138 (1941). [MF 6419]

This paper contains a theorem on the existence of functions $y_i(x)$ satisfying the system of differential equations

$$dy_i/dx=f_i(x, y_1, \cdots, y_n)$$

on the fixed interval (a, b), and the boundary conditions $y_i(a_i)=b_i$, where each a_i is a point of (a, b) and b_i is an arbitrarily fixed number. The functions f_i are supposed to be measurable in x and continuous in $(y_1, \cdots, y_n)$, and the solutions $y_i(x)$ are required to be absolutely continuous. The hypotheses involve a fairly complicated system of inequalities. The theorem is generalized to the case of a system of equations of higher order (the order of the derivatives need not be the same for all the dependent variables y_i), where the boundary conditions may also involve some of the derivatives. *L. M. Graves* (Chicago, Ill.).

Eachus, Joseph J. Classification of solutions and of pairs of solutions of $y'''+2py'+p'y=0$ by means of initial conditions. Duke Math. J. **9**, 20–24 (1942). [MF 6338]

This paper is concerned with the ordinary differential equation $y'''+2p(x)y'+p'(x)y=0$ under the following assumptions: (i) $p(x)$ is of class C' in the real variable x on the interval (a, b); (ii) there exists some solution of this differential equation possessing at least two double zeros on (a, b). The author determines the distribution and character of the zeros of an arbitrary solution, and also the relative distribution of the zeros of a pair of solutions of this equation. The classification is in terms of certain constants C_{ij}, C_i, D_{ij} defined as follows: C_{ij} is the constant value of $y_iy_j''+y_i''y_j-y_i'y_j'+2p(x)y_iy_j$ for two solutions y_i and y_j of this equation, $C_i=C_{ii}$ and $D_{ij}=C_{ij}-C_iC_j$. *W. T. Reid* (Chicago, Ill.).

Chaundy, T. W. Singular solutions of differential equations. I. Quart. J. Math., Oxford Ser. **12**, 129–147 (1941). [MF 5979]

For every positive integer n, differential equations of order n are constructed which have n distinct families of singular solutions, depending respectively on $n-1$, $n-2$, $\cdots$, 0 parameters. The equations are of generalized Clairaut types, with families of polynomials as general solutions; in one case the general solution is

$$-((n+1)!)^{-1}\sum_{i=1}^{n}(x+A_i)^{n+1},$$

where the A_i are arbitrary constants. The differential equations are obtained by a method, based on ingeniously constructed generating functions, which permits the singular solutions to be separated out by repeated differentiations. *J. F. Ritt* (New York, N. Y.).

Trjitzinsky, W. J. Developments in the analytic theory of algebraic differential equations. Acta Math. **73**, 1–85 (1941). [MF 5968]

The paper deals with the ordinary differential equations

(1) $$F(x, y, y', \cdots, y^{(n)})=0,$$

and

(2) $$F(x, \lambda, y, y', \cdots, y^{(n)})=0,$$

in which F is in each instance a homogeneous polynomial in $y, y', \cdots, y^{(n)}$, of degree $\nu>1$. In the case of the equation (1) the independent variable x is complex, and the coefficients are assumed to be actually or asymptotically representable by expressions $x^m\sum_{j=0}^{\infty}a_{-j}/x^j$ in some region R which extends to infinity. It is shown that if such an equation admits of a formal solution of the structure $s(x)=e^{Q(x)}x^r\sigma(x)$, with

$$Q(x)=h_0x^{p/k}+\cdots+h_{p-1}x^{1/k}, \quad h_0\neq 0; \ p>0,$$

and with $\sigma(x)$ a formal series in powers of $x^{-1/k}$, and if certain auxiliary hypotheses are fulfilled, then there exists

some infinite subregion R' of R within which some actual solution of the equation is analytic and fulfills, to any prescribed number of terms, the relations $y^{(i)}(x) \sim s^{(i)}(x)$, $i=0, 1, 2, \cdots, n$. In the case of the equation (2), λ is a complex parameter in a region R which extends to $\lambda=\infty$, while x is real and on an interval (a, b). The coefficients of the equation are taken to be indefinitely differentiable as to x, and to be representable in R by formal expressions $\lambda^m \sum_{j=0}^{\infty} \sigma_j(x)/\lambda^{j/k}$. It is shown that if such an equation admits of a formal solution of the structure $s(x, \lambda)=e^{Q(x,\lambda)}u(x, \lambda)$, with

$$Q(x, \lambda)=q_0(x)\lambda^{k/k}+q_1(x)\lambda^{(k-1)/k}+\cdots+q_{k-1}\lambda^{1/k},$$

and with $u(x, \lambda)$ a formal series $\sum_{j=0}^{\infty} u_j(x)/\lambda^{j/k}$, $k>0$, $q_0'(x) \not\equiv 0$, and $q_j(x)$, $u_j(x)$ indefinitely differentiable on (a, b), and if some auxiliary hypotheses are met, then there is some infinite subregion R' of R, and some subinterval (a_1, b_1) of (a, b), such that, for λ and x in these subdomains, some solution of the equation fulfills, to any specified number of terms, the relations $y^{(i)}(x, \lambda) \sim s^{(i)}(x, \lambda)$, $i=0, 1, 2, \cdots, n$. The subregions R', the subinterval (a_1, b_1) and the auxiliary hypotheses are determined in each case from theory associated with certain related linear differential equations. Their specifications are too intricate to permit of any brief summary here. *R. E. Langer* (Madison, Wis.).

Rellich, Franz. Elliptische Funktionen und die ganzen Lösungen von $y''=f(y)$. Math. Z. **47**, 153–160 (1940). [MF 6306]

If $P_3(y)$ is any polynomial of degree three with distinct roots, any solution of the equation $y'^2=P_3(y)$ which is analytic in a neighborhood of some point x_0 may be extended to exist as a meromorphic function over the entire complex x-plane, and is doubly periodic. This theorem, which is familiar from the theory of elliptic functions, is here very simply derived directly from the differential equation. It is similarly deduced that any two nonconstant meromorphic solutions $y_1(x)$, $y_2(x)$ of the equation are related thus: $y_2(x) \equiv y_1(x+c)$ with some constant c, and that, if $f(u)$ is a nonlinear entire function of u, every entire solution of the equation $u''=f(u)$ is a constant.

R. E. Langer (Madison, Wis.).

Erdélyi, A. The Fuchsian equation of second order with four singularities. Duke Math. J. **9**, 48–58 (1942). [MF 6341]

The solutions of Heun's differential equation are developed in series of hypergeometric functions. The domains of convergence of such expansions are regions whose boundaries are elliptic limaçons, each of which passes through one singularity of the differential equation and has two others as foci. These domains are ordinarily much more extensive than the circles of convergence of power series solutions. The new expansions have particular advantages when the differential equation admits of a solution which is regular at two or three of its singularities. In such cases the domains of convergence include enough of the equation's singularities to make the expansions useful in studies of the monodromic group. The method has applicability to differential equations with more singularities and to Fuchsian equations of higher orders. *R. E. Langer* (Madison, Wis.).

Brainerd, John Grist. Stability of oscillations in systems obeying Mathieu's equation. J. Franklin Inst. **233**, 135–142 (1942). [MF 6125]

Solutions of Mathieu's equation are classified in terms of the value of the parameter b or of the characteristic exponent μ, where $b=\cos 2\pi\mu$ and the notation is that used by Brainerd and Weygandt [Philos. Mag. (7) **30**, 458–477 (1940); these Rev. **2**, 285]. When $b<1$, μ is real and the solution is stable; when $b=1$, the solution is periodic of period 2π (the Mathieu function); and when $b>1$, μ is imaginary and the solution is unstable. The author supplies curves which define the domains of the parameters ϵ and k of the differential equation in which the solutions are of these different types; and also curves from which the actual value of μ may be obtained over an important part of the range of stable solutions. *M. C. Gray.*

Vessiot, Ernest. Sur la réductibilité des systèmes automorphes dont le groupe d'automorphie est un groupe continu fini simplement transitif. Ann. École Norm. (3) **57**, 1–60 (1940). [MF 4632]

This is an extension of the Picard-Vessiot theory of linear differential equations to automorphic systems. A system S of this type is defined as one whose manifold of solutions admits a transitive Lie group G, called the group of automorphisms of S. The equations of G may be taken to be $x_i'=\varphi_i(x_1, \cdots, x_n; a_1, \cdots, a_n)$, or $x'=\varphi(x, a)$, where these are also the equations of the first parameter group. If $\mathfrak{H}$ is the group of transformations $x'=\varphi(\theta(t), x)$, $t'=t$, where $\theta_i(t)$ are arbitrary functions of the t's and K is the subgroup of $\mathfrak{H}$ leaving S invariant, K is a second transitive group of transformations of the manifold of solutions; K is called the group of invariance of S. If the infinitesimal transformations of G are $\sum \xi_{ia}(x)\partial/\partial x_a$, the equations defining S may be taken in the form $\partial x_i/\partial t_k=\sum \lambda_{ka}(t)\xi_{ai}(x)$. The main part of the discussion concerns systems S for which the λ's and the ξ's are rational in the sense that they belong to a given differential field having prescribed properties. A subsystem Σ of S is called rational if it can be defined by the equations of S plus additional equations of the form $R(t, x)=0$, where R is in the field. A subsystem (rational or not) is called principal if it is automorphic relative to a subgroup g of G. Such a system has a group of invariance k which is a subgroup of K. On the other hand, if Σ is any subsystem and k is the largest subgroup leaving Σ invariant then there exist certain principal subsystems, called the adjoints of Σ, having k for group of invariance. Let k_0 be the subgroup of K leaving all rational subsystems invariant. Then it is shown that there exist principal rational subsystems with k_0 for group of invariance. A solution of such a system is called primitive, its transforms under k_0 the conjugates of u. The principal result, a type of Galois theorem, may now be stated: In order that a function which is rational in the x's and the t's reduce to a rational function of the t's when one replaces the x's by a primitive solution $(u_1, \cdots, u_n)$ of S, it is necessary and sufficient that it have the same value for all the conjugates of $(u_1, \cdots, u_n)$.

N. Jacobson (Chapel Hill, N. C.).

des Lauriers, Guérard. Sur les systèmes différentiels du second ordre qui admettent un groupe continu fini de transformations. Ann. École Norm. (3) **57**, 201–315 (1940). [MF 4635]

The author studies systems of n differential equations of second order which admit a Lie group of transformations. He remarks that a particular case of this problem is the problem of finding all the Riemannian spaces, the geodesic lines of which admit such a group; this problem has already been solved by the reviewer [Mem. Accad. Sci. Torino (2) **43** (1903); Atti Accad. Naz. Lincei. Rend. **14** (1905)].

The first chapter is concerned with two kinds of systems of differential equations. Systems I define the second derivatives of unknown functions x_i $(i=1, 2, \cdots, n)$ with respect to a parameter t as functions of the x_j and their first derivatives (t not appearing explicitly in the equations). Systems II give d^2x_i/dx_n^2 $(i=1, 2, \cdots, n-1)$ as functions of the x_j and of the dx_j/dx_n. Given two systems I, the author derives conditions under which it is possible to transform one into the other by means of a change of the independent variable t, and similarly conditions that a system I be equivalent to a system II. He finds that a special rôle is played by the systems III of n equations:

$$\frac{d^2x_i}{dt^2}=\sum_{r,s} f_{rs}^i \frac{dx_r}{dt}\frac{dx_s}{dt}+\frac{dx_i}{dt}R,$$

where the f_{rs}^i are functions of the x_j only and R is independent of i and a function of the x_j and dx_j/dt. For such systems the author generalizes Riemann's symbols of the second kind and Ricci's calculus in Riemannian spaces.

Next conditions are stated under which a given system III admits an infinitesimal transformation

$$X=\sum_i \xi^i \frac{\partial}{\partial x_i}, \quad \xi^i \text{ functions of the } x_j,$$

and the corresponding partial equations for the ξ are found. Using the conditions of integrability for these equations, the author shows that the second derivatives of the ξ are linear functions of the ξ and their first partial derivatives with respect to the x_j. There exist special systems which, by a suitable choice of the x_j and of the independent variable, can be transformed into the system $x_i''=0$. This case excepted, the author finds the maximum number M of parameters on which a group transforming a system III into itself may depend. For example, if $n\geqq 3$, then $M=(n-1)(n-2)+3$. In studying the infinitesimal transformations of a group Γ which transforms a system III into itself, he takes into account their order r; if Taylor's development of the ξ begins with terms of degree r, the transformation X is of order r. From the preceding theorems it follows that, in the most important cases, r is equal either to zero or to 1. The following investigation concerns the set of transformations of Γ of order zero, and the set γ of transformations of order 1 (which transforms the origin into itself), and the relations among such transformations. Moreover, a study is made of the group $\bar{\gamma}$ generated by the infinitesimal transformations of γ, where all the terms which are not of first order in the development of the ξ are omitted. Owing to the long calculations a detailed description of them is impossible. Next the author studies systems having a group Γ depending on M or $M-1$ parameters; here the properties of the corresponding group $\bar{\gamma}$ are important. Despite their length the calculations in the paper are not complete. Finally the author determines all the systems of three equations which admit a group. Many results are summarized in synoptic tables.

G. Fubini (Princeton, N. J.).

***Gantmacher, F. R. and Krein, M. G. Oscillation matrices and small oscillations of mechanical systems.** Moscow-Leningrad, 1941. (Russian)

The authors are interested in applications of the theory of matrices and vectors (and more generally, of the theory of linear integral equations) to the theory of small oscillations of mechanical systems. It turns out that the matrices which they call "oscillation matrices" are particularly useful and possess very interesting properties. The definition of such matrices can be formulated as follows. The square matrix $A=\|a_{ik}\|_1^n$ is called of definite sign if for each $p=1, 2, \cdots, n$ all minors of A of order p distinct from zero have the same sign ϵ_p. If in addition all minors are distinct from zero, A is said to be strictly of definite sign. The matrix A is called completely nonnegative (completely positive) if all minors of A are nonnegative (positive). If A is completely nonnegative and there is a positive integer k such that A^k is completely positive, then A is called an oscillation matrix. In order that a completely nonnegative matrix A be an oscillation matrix it is necessary and sufficient that $\operatorname{Det} A\neq 0$ and $a_{i,i+1}$, $a_{i+1,i}>0$ for $i=1, 2, \cdots, n-1$. The characteristic values of an oscillation matrix are all simple and positive.

Chapter 1 of the book furnishes a general introduction in the theory of matrices and quadratic forms. In chapter 2 the authors give a detailed discussion of various matrices (of definite sign, completely nonnegative, completely positive, and oscillation matrices), of their characteristic values and characteristic vectors. The results obtained in chapter 2 are applied in chapter 3 to the theory of small oscillations of mechanical systems. For lack of space details cannot be stated here. Finally, in an appendix they sketch how the theory could be applied to the theory of integral equations, essentially of the type studied by O. Kellogg.

J. D. Tamarkin (Providence, R. I.).

Ingram, W. H. Forced oscillations of continuous dynamical systems. Bull. Amer. Math. Soc. **48**, 153–162 (1942). [MF 6196]

This paper is concerned with the solution of the system of differential equations

$$(*) \qquad \partial u_i/\partial x=\sum a_{ij}(x)\partial u_j/\partial t+\sum b_{ij}(x)u_j+e_i(x, t)$$

under the boundary conditions

$$\sum\{\alpha_{ij}u_j(a)+\beta_{ij}u_j(b)\}=0.$$

For $e_i(x, t)\equiv 0$ and with the substitution $u_i(x, t)=e^{st}y_i(x)$, (*) may be reduced to the system of ordinary differential equations written in matrix form:

$$(**) \qquad Y'(x)=(\mu A+B)Y, \quad W_aY(a)+W_bY(b)=0,$$

where $A=(a_{ij})$, $B=(b_{ij})$, $W_a=(\alpha_{ij})$, $W_b=(\beta_{ij})$ are $n\times n$ matrices and Y is a columnar matrix of n elements. For e_i in (*) for which the Burkhardt transform [Bull. Soc. Math. France **22**, 71 (1894)] can be represented by a linear combination of characteristic vector solutions of (**) assumed to have a Green's matrix, a solution is given for the Fourier-Bernoulli problem as well to the Heaviside problem for (*).

A. E. Heins (Cambridge, Mass.).

Picone, Mauro. Su una recensione della mia Memoria: Nuovi metodi per il calcolo delle soluzioni delle equazioni a derivate parziali della Fisica-Matematica degli "Annali Scientifici" dell'Università di Jassy, Vol. XXVI, Sez. I (1940). Boll. Un. Mat. Ital. (2) **3**, 273–280 (1941). [MF 5596]

A reply to the review by R. V. Churchill in these Rev. **1**, 236. The author objects to the statements (a) that the author's method of solving differential equations is complicated and (b) that the choice of other than a polynomial for the approximate solution would be impractical. The author states that complexity is not the fault of his method but of the generality of the problems treated. In stressing

the generality of the method the following problems are cited as examples which can be reduced to numerical solutions: (I) a general system of three linear second order partial differential equations with variable coefficients inside a cube, with general systems of linear first order equations with variable coefficients as boundary conditions on the faces; (II) the Dirichlet problem for the region between two concentric and homothetic squares; (III) one of a certain class of fourth order equations with constant coefficients in a square. An example is given in reply to (b).
P. W. Ketchum (Urbana, Ill.).

***Carslaw, H. S. and Jaeger, J. C. Operational Methods in Applied Mathematics.** Oxford University Press, New York, 1941. xvi+264 pp. $5.00.

The connection between the Laplace transformation and Heaviside's operational calculus is discussed in a short historical introduction. Several basic properties of the transformation are derived in chapter 1. Their use in solving systems of linear ordinary differential equations with constant coefficients is illustrated there and in the next two chapters on electric circuits and applications to dynamics. The analysis in these first three chapters is kept to the level of calculus. The other seven chapters are more advanced. The inversion of the Laplace transformation by the usual integral in the complex plane, the evaluation of this integral by the calculus of residues and contour integration, and applications to the solution of linear boundary value problems in partial differential equations are introduced in chapters 4 and 5. The remaining chapters present the solutions of an interesting variety of problems in partial differential equations in various branches of mathematical physics: conduction of heat, vibrations, hydrodynamics, electric transmission lines and electromagnetic fields. The book contains a large number of exercises in ordinary and partial differential equations. *R. V. Churchill.*

***McLachlan, N. W. et Humbert, Pierre. Formulaire pour le calcul symbolique.** Mémor. Sci. Math., fasc. 100, 67 pp. (1941). 25 fr.

The lack of an extensive list of Laplace transforms (L.-t.) was felt badly by many workers in the field. The authors have undertaken the task of collecting all known explicit formulas for L.-t. $\phi(p)=p\int_0^\infty e^{-pt}f(t)dt$, and have added a few new ones. The result is almost 700 formulas collected in the present excellent pamphlet. First there are about 40 "general" formulas: here we find, for example, the L.-t. of

$$(t\pi^{\frac{1}{2}}/2)\int_0^\infty J_{\frac{1}{4}}[3(t^3x/4)^{\frac{1}{2}}]f(x)x^{-\frac{1}{4}}dx$$

(namely $\phi(p^{-2})$), of $t^nf(t^2)$ and of similar expressions. The "special" formulas are collected under headings which are to be understood in the broadest sense: for example, under "hyperbolic functions" we find the L.-t. of

$$t^{-\frac{1}{2}}\{\mathrm{ch}\ (8t)^{\frac{1}{2}}-\cos\ (8t)^{\frac{1}{2}}\}.$$

The following are the chapter headings, the figures in parentheses indicating, approximately, the number of formulas under the respective headings: algebraic functions (19); exponentials and logarithms (40); circular functions (46); hyperbolic functions (59); integral logarithms (10); error functions (16); Bessel functions (68); Bessel functions of second kind (15); Hankel functions (26); Bessel functions of an imaginary argument (60); Bessel's k-function (47); Kelvin's functions ber and bei (20); ker and kei (18); Struve functions (23); Struve functions of an imaginary argument (22); Bessel functions of third order (15); confluent hypergeometric functions (12); Whittaker functions (11); Weber functions (20); hypergeometric series (11); polynomials of hypergeometric type (48); discontinuous functions (27); L.-t. in two variables (23). *W. Feller.*

Russell, J. B. Heaviside's direct operational calculus. Elec. Engrg. 61, 84–88 (1942).

This is the first of a group of five lectures given before the American Institute of Electrical Engineers as a symposium on "Advanced Mathematics as Applied to Electrical Engineering."

Cromwell, Paul C. A construction theorem for evaluating operational expressions having a finite number of different roots. Trans. Amer. Inst. Elec. Engrs. 60, 273–276 (1941).

A known method for finding particular integrals of linear constant-coefficient ordinary differential equations is offered as an alternative to the usual Cauchy-Heaviside partial-fraction expansion approach. As presented and applied, the alternative method lacks the simplicity and generality (the equal root case is excluded; operand must be made a polynomial) of the usual procedure. The method is formally applied to the evaluation of products of irrational binomial operators with unit-step-function operand to obtain asymptotic results. By using the integral inverse Laplace transformation an indication is given of how the extension could be justified. *J. L. Barnes* (Medford, Mass.).

Azevedo do Amaral, Ignacio M. On the integration of ordinary linear differential equations and integral equations. Anais Acad. Brasil. Sci. 13, 305–317 (1941). (Portuguese) [MF 6588]

This paper studies the application of the transformation of Laplace

$$y(x)=\int_a^b e^{\alpha x}u(\alpha)d\alpha$$

to ordinary linear differential equations of special type.
A. E. Heins (Cambridge, Mass.).

Azevedo do Amaral, Ignacio M. On the integration of ordinary linear differential equations. Anais Acad. Brasil. Sci. 12, 191–196 (1940). (Portuguese) [MF 6574]

The author suggests the use of the transformation

$$y(x)=\int_a^\infty e^{-\alpha x}\phi(\alpha)d\alpha$$

and its inverse for the solution of a linear differential equation of the type

$$\sum_{i=0}^{n} X_i y^{(n-i)}=F(x),$$

where $F(x)$ and X_i are prescribed functions of x. Conditions of validity are not given and it is not clear from the text of the paper to what forms of X_i and $F(x)$ the above transformation may be applied. *A. E. Heins.*

Bartels, R. C. F. and Churchill, R. V. Resolution of boundary problems by the use of a generalized convolution. Bull. Amer. Math. Soc. 48, 276–282 (1942). [MF 6406]

The authors define the integral from 0 to t of $F(t-t', t')$ with respect to t' as the generalized convolution of $F(t, t')$

and denote it by $F^*(t)$. Under certain restrictions on $F(t, t')$,

$$(*) \qquad \int_0^\infty e^{-st}F^*(t)dt = \int_0^\infty e^{-st'}dt' \int_0^\infty e^{-st}F(t, t')dt.$$

The left side of the above equation is the Laplace transform of $F^*(t)$, and the right side is an iterated transform of $F(t, t')$. These two points of view of (*) enable the authors to replace temperature problems with variable source and surface conditions by temperature problems with constant source and surface conditions. A formal application of (*) is also made to vibration problems.

F. G. Dressel (Durham, N. C.).

Bleuler, Konrad. Sur la variation d'un corps à potentiel stationnaire. C. R. Soc. Phys. Genève **58**, 220–221 (1941). [MF 6506]

Let S be a region of the plane. The author states, without proof, a necessary and sufficient condition that a sequence of closed contours in S (depending analytically on a parameter α) bound regions for which the logarithmic potential

$$V(\alpha, P) = \rho(\alpha) \int\int_{S_\alpha} \log r_{PP'} dS_\alpha(P')$$

is independent of α for all points P exterior to S_α.

F. W. Perkins (Hanover, N. H.).

Whitehead, S. An approximate solution for the distribution of temperature or potential with cylindrical isothermal or equipotential surfaces. Proc. Phys. Soc. **54**, 63–65 (1942). [MF 6230]

This paper considers an approximate method for calculating the field between several equipotential (isothermal) cylinders. An example dealing with three cylinders is considered and a graphical treatment is indicated. Special configurations are discussed. *A. E. Heins.*

Keck, W. G. and Colby, W. F. The depth dependence of earth conductivity upon surface potential data. J. Appl. Phys. **13**, 179–188 (1942).

The problem of determining the conductivity (assumed to be a function of depth only) from surface potential data with a single point-electrode is considered. A perturbation method is used, in which the zero-approximation is the solution (supposed known) for an arbitrarily assigned conductivity. The first order perturbation then leads to an integral equation of the first kind for the determination of the actual unknown conductivity. Two cases are considered: (1) the "zero-order" conductivity is uniform (a case previously considered by Stevenson); (2) the zero-order conductivity varies exponentially with depth. Case (2) is more general than case (1), and is considered in detail. It leads to an integral equation of the Laplace type, for which an approximate method of numerical solution is given. A rough rule for determining the best value of the arbitrary parameter occurring in the zero-order conductivity is also given. Two numerical examples, where the accurate solution is known, are worked out as a test of the method, with satisfactory results. In one of these, the numerical calculations are modified in such a way as to allow for a variation of the parameter in the zero-order conductivity.

A. F. Stevenson (Toronto, Ont.).

Trjitzinsky, W. J. Analytic theory of singular elliptic partial differential equations. Ann. of Math. (2) **43**, 1–55 (1942). [MF 6058]

The subject of discussion is the partial differential equation

$$F(u) \equiv \sum_{i,k=1}^{m} A_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{i=1}^{m} B_i \frac{\partial u}{\partial x_i} + Cu = f,$$

with $m > 2$, with $A_{ik} \equiv A_{ki}$, and with A_{ik}, B_i, C and f functions of $x = (x_1, \cdots, x_m)$, in an open domain D of Euclidean m-space. It is assumed that $f \in L_2$, that the B_i are differentiable and that the A_{ik} have continuous derivatives up to order four in D. Using the operator adjoint to F, the equation is transformed into an integral equation which fulfills certain defined conditions of regularity. The considerations are then focused upon the case of an equation which depends upon a complex parameter λ. Specifically, the equation $F(u) + \lambda u = f$ (in D_n) with $\partial u/\partial v + a_n(x) = 0$ (on S_n, the limiting surface of D_n), where D_n and S_n are in a certain sense regular, and $\lim D_n = D$, are considered. There exist characteristic values and solutions, and the latter together with those of the adjoint problem fulfill a relation of biorthogonality. The discussion is then restricted to the self-adjoint case. The matters discussed include the existence of solutions contained in L_2 when λ is in certain defined sets of the complex plane, a spectral theory, generalized Fourier expansions, etc. The paper is professedly modeled upon, and is in the main a generalization of, work by Carleman on the Schrödinger wave equation [Ark. Mat. Astr. Fys. **24B**, no. 11, 1–7 (1934)], in which $m = 3$, $B_i \equiv 0$ and $A_{ik} \equiv 0$ $(i \neq k)$, $A_{ii} = 1$. *R. E. Langer* (Madison, Wis.).

Minakshisundaram, S. On the expansion of an arbitrary function in a series of eigenfunctions of boundary value problems. II. J. Indian Math. Soc. (N.S.) **5**, 103–108 (1941). [MF 6126]

Let D be a domain of the plane, simply or multiply connected, whose boundary C consists of a finite number of regular curves. Consider the boundary value problem:

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \lambda\omega = 0, \qquad \omega(x, y) = 0 \text{ on } C.$$

Let $\omega_n(x, y)$ be the eigenfunctions (normal and orthogonal) with the corresponding eigenvalues $\lambda_n > 0$, $n = 1, 2, 3, \cdots$. Corresponding to a function $f(x, y)$ integrable in D, we have the formal expansion $f(x, y) \sim \sum a_n \omega_n(x, y)$, where

$$a_n = \int\int_D f(x, y)\omega_n(x, y)dxdy.$$

The paper contains a discussion of the following theorem: If the series $\sum \lambda_n a_n^2$ converges, then at any interior point of D at which $f(x, y)$ is continuous the series $\sum a_n \omega_n(x, y)$ converges to $f(x, y)$. The author also discusses a corresponding theorem for space of three dimensions; here the series $\sum \lambda_n a_n^2$ is replaced by $\sum \lambda_n^{\frac{1}{2}} a_n^2$. The author states, without proof, the conjecture that a corresponding theorem is valid in a space of m dimensions, provided that the series $\sum \lambda_n a_n^2$ is replaced by $\sum \lambda_n^{m/2} a_n^2$. *F. W. Perkins.*

Weinstein, Alexander. On the decomposition of a Hilbert space by its harmonic subspace. Amer. J. Math. **63**, 615–618 (1941). [MF 4686]

The Hilbert space H of all functions of integrable square defined in a sphere S of 3-space contains the subspace $\mathfrak{P}$ of all harmonic functions regular in S and its orthogonal

complement $\mathfrak{Q}$. The author and N. Aronszajn investigated the sets $\mathfrak{P}$ and $\mathfrak{Q}$ recently [Proc. Nat. Acad. Sci. U. S. A. 27, 188–191 (1941); cf. these Rev. 3, 44] and obtained results which lead to a proof of a lemma used by H. Weyl in a recent article [Duke Math. J. 7, 414–444 (1940); cf. these Rev. 2, 202]. Let C be the boundary of the sphere S and let K_2 be the class of all continuous functions ζ possessing continuous first and second derivatives in $S+C$ and vanishing along with these derivatives on C. Then Weyl's lemma states that a function η in H satisfying the equation $(\eta, \Delta\zeta)=0$ for every ζ in K_2 is harmonic. The author's proof is based on the use of a complete sequence of functions in $\mathfrak{Q}$ and orthogonal to η. *H. H. Goldstine.*

Schrödinger, E. Further studies on solving eigenvalue problems by factorization. Proc. Roy. Irish Acad. Sect. A. 46, 183–206 (1941). [MF 5659]

The author continues the study of his method of solving eigenvalue problems by factorization of the differential operator [same Proc. 46, 9–16 (1940); these Rev. 1, 277]. The general form of the required factorization (which is not unique) is obtained, and the effect of using different factorizations is illustrated in the case of the problem of s identical harmonic oscillators. When set up in polar coordinates, the amplitude equation for the radius vector in this problem turns out to be related to the radial equation for the Kepler problem. The question of proving, for a particular problem, that the author's "manufacturing process" produces all eigenfunctions of the discrete spectrum is considered. It is indicated that the proof will usually follow from the fact that the eigenvalues have a lower bound, and from the fact that the first order operators involved change admissible functions into admissible functions, but the details of the proof will vary with different problems, and also with different factorizations. The generality of the factorization method is shown by applying it to a problem with artificial boundary conditions of a sort not usually encountered in quantum mechanics. For the case of a pair of first order equations, a variation of the original method is developed which consists of transforming the variables so that the new first order operators can themselves be used to manufacture all the eigenfunctions from a single one, without performing a factorization.

Finally the author presents a new perturbation theory based on his method of factorization. It consists of supplying the first order factors of the unperturbed operator with small additional terms so as to produce approximately the perturbed term of the Hamiltonian, thus giving an approximate factorization of the perturbed operator. The method is applied to the well-known case of the Stark effect for the hydrogen atom. When applicable, the new method has the advantage of giving the perturbed eigenvalues, and the perturbed eigenfunctions in finite form, without quadratures. *O. Frink* (State College, Pa.).

Schrödinger, Erwin. On the solutions of wave equations for non-vanishing rest-mass including a source-function. Proc. Roy. Irish Acad. Sect. A. 47, 1–23 (1941). [MF 5872]

The author proposes the study, somewhat along the lines of electromagnetic theory, of the generalized wave equation (1) $(\nabla^2-\partial^2/\partial t^2-\mu^2)U=f$, as a possible basis for a new theory of nuclear forces. In this equation the source function f, which depends on the coordinates and time, describes the distribution of the heavy particles, and μ is a constant determined by the rest-mass of the light-weight particles, of which U is the wave function. If $f=0$, equation (1) reduces to the de Broglie equation, and if $\mu=0$, it reduces to the Maxwell equation with a source function. Replacing U and f by their spherical mean values and transforming variables reduces (1) to the one-dimensional case (2) $(\partial^2/\partial r^2-\partial^2/\partial t^2-\mu^2)\chi=q$. Three particular solutions of the homogeneous case ($q=0$) of (2) are derived, an exponential de Broglie wave and the two Bessel function solutions $I_0(\mu s)$ and $I_1(\mu s)/s$, where s is $(t^2-r^2)^{\frac{1}{2}}$. The method of the Green's function is then used to express the general solution of the initial value problem for (2), and hence the general spherically symmetric solution of this problem for (1), in terms of these particular solutions. For the static case where f is independent of the time, it is shown that solving (1) gives the ordinary static potential formula with $1/r$ replaced by the Yukawa potential $\exp(-\mu r)/r$, thus providing a link with other nuclear theories.

These results are then applied to the problem of finding the field associated with a point-like charge in translatory motion. The moving charge is described by a four-vector (f), the density of current and charge, and, as in electromagnetic theory, the field is to be described by a vector potential (U), which is to be obtained from (f) by means of equation (1). The author uses the device of considering (U), or one of its components, to be a solution of the homogeneous case ($f=0$) of equation (1), with prescribed singularities matching the source function. This allows him to obtain solutions by integrating with respect to the parameter a known one-parameter family of solutions. Expressions for the components of (U) are obtained in this way by complex integration with respect to a complex parameter of a family of solutions similar to those obtained earlier, but involving the Neumann function solutions $N_0(\mu s)$ and $N_1(\mu s)$ of the Bessel equation, which have singularities for $s=0$ (that is, on the light-cone), rather than the ordinary Bessel functions. *O. Frink* (State College, Pa.).

Faedo, Sandro. Contributo alla sistemazione teorica del metodo variazionale per l'analisi dei problemi di propagazione. Ann. Scuola Norm. Super. Pisa (2) 10, 139–152 (1941). [MF 6420]

Let (1) $p^{ij}_{\sigma\rho}u^\sigma_{ij}+p^{i}_{\sigma\rho}u^\sigma_i+p_{\sigma\rho}u^\sigma-p_\rho=E_\rho=0$, where $\rho, \sigma=1, \cdots, 3$; $i, j=1, \cdots, 4$; the independent variables are $x_1, x_2, x_3, x_4=t$; $p^{44}_{\sigma\rho}=0$ unless $i=4$, subscripts on u^σ indicate the obvious derivatives and dummy indices are summed. Let K be the unit cube and to each face F associate the boundary condition (2) $D_{\rho,F}=(a^i_{\sigma\rho}u^\sigma_i+a_{\sigma\rho}u^\sigma-a_\rho)|_F=0$. The p's and a's are of class C. Initial conditions are prescribed values of $u^\sigma|_{t=0}$, $u^\sigma_t|_{t=0}$ in C^2. Let the νth approximation [cf. (6) below] be

$$(4)\qquad u^{\sigma,\nu}=u^\sigma|_0+tu^\sigma_t|_0+\sum_{n=1}^{\nu}\phi^n(x_1, x_2, x_3)\omega_n{}^\sigma(t),$$

where the ϕ's are known and in C, and the ω's are to be determined subject to absolute continuity on all finite intervals for $\omega_n{}^\sigma$ and $(d/dt)\omega_n{}^\sigma$ and $\omega^\sigma(0)=d\omega^\sigma(0)/dt=0$. Write $W^\nu=\{\omega_i{}^\sigma\}$ and

$$(5)\qquad F^\nu=F(t; W^\nu; dW^\nu/dt)=\int_K q^\sigma(E_\sigma(u^{\sigma,\nu}))^2+\int_F q^{\rho,F}(D_{\rho,F}(u^{\sigma,\nu}))^2,$$

where $\{q^\sigma\}$, $\{q^{\rho,F}\}$ are arbitrary positive continuous weight functions of class C. Finally write (6) $I^\nu=\int_0^\alpha F^\nu dt$. Throughout this paper ν is fixed. For finite t intervals the existence of a minimum for I^ν is known. The writer indicates that

necessary and sufficient conditions for $\bar{u}^{\sigma,\nu}$ to minimize I^ν are that for a particular W^ν the components ω_i^σ as well as the $d\omega_i^\sigma/dt$ are absolutely continuous on all finite t intervals and the associated I^ν is finite.

For the sufficiency proof a Cantor diagonal principle argument leads from the known existence of admissible convergent minimal W^ν sequences for $0 \leqq t \leqq N, N=1,2,\cdots$, to the existence of a sequence convergent on any finite interval. If q^σ, $q^{\sigma,\nu}$ depend on t alone, then a sufficiency condition is that each of the integrals in the implied sum on the right of (5) belongs to $L(0,\infty)$ when $\omega_n^\sigma = 0$. Since q^σ, $q^{\sigma,\nu}$ can always be found to yield such a condition, there is always a minimizing $\bar{u}^{\sigma,\nu}$ (for proper q's). The best ω_n^σ's for (6) may be determined from the Euler equation or an integral equivalent valid when $\omega_n^\sigma \in C^2$. It is not shown that the best approximations $\bar{u}^{\sigma,\nu}$, $\nu=1,2,\cdots$, converge in any sense to a solution (or generalized solution) of (1), (2) and (3) and indeed, because of the dependence on the q's, the relation of $\bar{u}^{\sigma,\nu}$ to a solution needs interpretation.

D. G. Bourgin (Urbana, Ill.).

Picone, Mauro. Nota al precedente lavoro. Ann. Scuola Norm. Super. Pisa (2) **10**, 153–155 (1941). [MF 6421]

Let $\theta(t)$, $\alpha(t)$, $\beta(t)$, $\rho(t) \in C(0,\infty)$ with $\theta^2\rho > 0$. Let (A) $\beta\theta/\alpha \in L_2(0,\infty)$, then (B): $\omega'' + 2\theta'\omega'/\theta - \alpha^2\omega = \beta$, $\omega(0)=0$ has a solution $\tilde{\omega}$ such that (D) $\theta\alpha\tilde{\omega}$ and $\theta\tilde{\omega}' \in L_2(0,\infty)$. The details are implicit in Picone's exposition, but may be made clear as follows. In the previous paper assume u, x, t are the only variables and u_{tt} is absent from (1). Then ω_n in (4) satisfies only one condition, namely $\omega_n(0)=0$. By the second sufficiency condition in the previous paper I^1, with $F^1 = \omega'^2 + \alpha^2\omega^2 + 2\beta\omega + (\beta/\alpha)^2 + \rho$, has a minimum for $\omega = \tilde{\omega}$ if $q = \theta^2$ satisfies $\theta\beta/\alpha$, $\rho^{\frac{1}{2}}\theta \in L_2$, and since ρ is absent from (1) we arbitrarily select a satisfactory ρ. Thus $\theta^2 F^1(t, \tilde{\omega}, \tilde{\omega}') \in L_2$, and by combining with (A) we get (D). Finally $\tilde{\omega}$ satisfies the Euler equation derived from I^1 and this is precisely (B).

D. G. Bourgin (Urbana, Ill.).

Hölder, Ernst. Bemerkung zu Riemanns Abhandlung "Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite." Abh. Math. Sem. Hansischen Univ. **14**, 338–350 (1941). [MF 5450]

The equation of motion and the equation of continuity of a one-dimensional motion of a compressible fluid of velocity $u(x,t)$ and of density ρ can be written in the following form:

$$\frac{\partial u}{\partial t} = \frac{u}{c}\frac{\partial v}{\partial t} - \frac{c^2-u^2}{c}\frac{\partial v}{\partial x}, \qquad \frac{\partial u}{\partial x} = -\frac{1}{c}\frac{\partial v}{\partial t} - \frac{u}{c}\frac{\partial v}{\partial x},$$

where $v(\rho) = \int c\, d\log\rho$, $c = dp/d\rho > 0$, p denoting the pressure. These equations are the Cauchy-Riemann-Beltrami equations corresponding to the indefinite $ds^2 = (c^2-u^2)dt^2 + 2u\,dt\,dx - dx^2$. The characteristics $x = x_1(t,\alpha)$ and $x = x_2(t,\beta)$ are given by $ds^2 = 0$. The author gives a discussion of the initial value problem: u given for $t=0$, using α and β as the new independent variables [method of H. Lewy, Math. Ann. **98**, 179 (1928)]. *A. Weinstein* (Toronto, Ont.).

Avazašvili, D. Z. Theorem of uniqueness of solution of electromagnetic Maxwell equations in a nonhomogeneous infinite medium. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **8**, 109–134 (1940). (Russian. Georgian summary) [MF 5312]

Let $\mathbf{F}(x,y,z)e^{-i\omega t}$ be a given electromagnetic monochromatic wave in the 3 dimensional space. Consider a body T_i obstructing the wave propagation. Let T_a be the domain exterior to T_i. The author proves the uniqueness of the solution of the following diffraction problem. [Standard notations are used throughout.] To find an electric field $\mathbf{E}$ and a magnetic field $\mathbf{H}$ satisfying the equations $\Delta\mathbf{E} + s^2\mathbf{E} = 0$, $\Delta\mathbf{H} + s^2\mathbf{H} = 0$, $\operatorname{div}\mathbf{E} = \operatorname{div}\mathbf{H} = 0$ ($c^2s^2 = \omega^2\epsilon_i\mu_i + i\omega\mu_i\sigma_i$ in T_i; $c^2s^2 = \omega^2\epsilon_a\mu_a + i\omega\mu_a\sigma_a$ in T_a), $i\omega\mu\mathbf{H} = c\operatorname{curl}\mathbf{E}$; $(\sigma - i\omega\epsilon)\mathbf{E} + \sigma\mathbf{F} = c\operatorname{curl}\mathbf{H}$; $E_{ti} = E_{ta}$; $H_{ti} = H_{ta}$; $\nu_i E_{ni} = \nu_a E_{na}$ ($\nu = \sigma - i\omega\epsilon$); $\mu_i H_{ni} = \mu_a H_{na}$; $\mathbf{E} = O(R^{-1})$; $\mathbf{H} = O(R^{-1})$; $\partial\mathbf{E}/\partial R = \partial\mathbf{H}/\partial R = O(R^{-1})$; $\lim R(\partial\mathbf{E}/\partial R - is\mathbf{E}) = \lim R(\partial\mathbf{E}/\partial R - is\mathbf{H}) = 0$ for $R \to \infty$. The author follows the method used by H. Freudenthal [Compositio Math. **6**, 221–227 (1938)] in the two dimensional case. *A. Weinstein* (Toronto, Ont.).

Green, George. Corresponding problems in periodic and steady flow. Philos. Mag. (7) **33**, 102–114 (1942). [MF 6256]

The idea of this paper is that some solutions of (1) $\nabla^2 w = 0$ may be obtained from the periodic solutions of (2) $\partial^n v/\partial t^n - \nabla^2 v = 0$, $n=1,2$, by

$$(3) \qquad w = L_{t\to\infty}\int_0^t\int_0^\infty v(k; t-T; x, y, z)\,dk\,dT.$$

For instance, v may be $(\exp ik(t-T) - (ik)^{\frac{1}{2}})/r$. This idea is formally applied to a large number of special cases and yields correct w values when "infinite constants" are neglected. If the series expansion of v contains derivatives of functions vanishing at ∞ then such terms make no contribution to w (assuming all manipulations are legitimate). Perhaps some minor interest may attach to a rigorous validation of the process of going from (2) to (1), analogous to the Hadamard method of descent for hyperbolic equations; however, the author's applications of (3) indicate merely an involved heuristic procedure for obtaining obvious results. Thus v is often a complicated expansion in terms of products of Bessel and Legendre functions, while w is never worse than A/r or $A\log r$. *D. G. Bourgin* (Urbana, Ill.).

Krzyżański, Miroslaw. Sur les solutions des équations du type parabolique déterminées dans une région illimitée. Bull. Amer. Math. Soc. **47**, 911–915 (1941). [MF 5936]

A function $g(x_1,\cdots,x_n) = g(x)$ will be said to belong to class E if it is continuous and there exist two positive constants M and K such that $|g(x)| < M\exp(K\sum_{i=1}^n x_i^2)$. Let R be the infinite region: $0 \leqq y \leqq h$, $-\infty < x_i < \infty$ $(i=1,\cdots,n)$, and consider the parabolic equation

$$(*) \qquad \sum_{i,j=1}^n a_{ij}(x,y)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x,y)\frac{\partial u}{\partial x_i} + a(x,y)u - \frac{\partial u}{\partial y} = 0,$$

where $\sum_{i,j=1}^n a_{ij}\lambda_i\lambda_j$ is a positive definite form. The paper shows that, if the coefficients of (*) are continuous and bounded in R, the only solution of the above parabolic equation (*) which belongs to class E in R and vanishes on the characteristic $y=0$ is $u=0$. If, in addition to the above assumptions on the coefficients of (*), we assume the a_{ij} have continuous first derivatives, the author states that for $\phi(x)$ belonging to class E there exists one and only one solution of (*) belonging to class E and assuming the value $\phi(x)$ on the characteristic $y=0$. The proof of this result must be looked upon as incomplete, since it is based on certain statements of G. Giraud and M. Gevrey. Under the continuity conditions of the present paper on the a_{ij}, a_i and a, these statements of Giraud and Gevrey are not in reference to the equation (*) but to what might be called a "gener-

alized" parabolic operator. The definition of the function $H(x, y)$ in (4), page 912 needs several modifications to produce the inequality at the top of page 913. *F. G. Dressel.*

Koschmieder, Lothar. Beziehungen zwischen Temperaturen und Potentialen. Math. Z. **47**, 125–131 (1940). [MF 6303]

Some of the integral compositions of solutions of partial differential equations in two variables discussed by Doetsch [Math. Z. **46**, 315–328 (1940); these Rev. **1**, 314] are generalized here to any number of variables and functions. A composition of n solutions of simple heat equations that represents a solution of the n-dimensional Laplace equation is given first. This is used to obtain a singular solution of Laplace's equation. Integral compositions of several solutions of parabolic equations in several variables, and of one such solution with a potential function of several variables, are also shown to satisfy certain differential equations. *R. V. Churchill* (Ann Arbor, Mich.).

Jaeger, J. C. Conduction of heat in regions bounded by planes and cylinders. Bull. Amer. Math. Soc. **47**, 734–741 (1941). [MF 5477]

The first problem solved here is that of finding the temperature $u(r, z, t)$ in the cylindrical region $0 \leqq r < a$, $z > 0$, when $u(a, z, t) = 1$ and $u(r, 0, t) = u(r, z, 0) = 0$. The method employed is a combination of the Laplace transformation with respect to t and the use of series of particular solutions of the transformed equation. In this case the author is able to express the function u in a form well suited to computation, a series involving the error function and Bessel functions. The same method is used here to obtain the temperature distributions, under similar boundary conditions, in the region $r > a$, $0 < z < l$, and in the region $r > a$, $z > 0$, and also in the finite cylinder. In each problem the initial temperature is taken as zero, one of the boundary surfaces is kept at unit temperature and the others at temperature zero. *R. V. Churchill* (Ann Arbor, Mich.).

Jaeger, J. C. Heat conduction in composite circular cylinders. Philos. Mag. (7) **32**, 324–335 (1941). [MF 5914]

The author finds the solution of the following problems in heat conduction with the aid of the Laplace transform [cf. these Rev. **1**, 223; **2**, 56, 204, 292]: (a) Hollow cylinder of one material from $r = a$ to $r = b$ and another from $r = b$ to $r = c$. The initial temperature of the whole body is zero. The surface $r = a$ kept at a temperature V_0 and $r = c$ at zero temperature, for t(time) > 0. (b) Same body as in (a). The surfaces $r = a$ and $r = c$ kept at zero temperature for $t > 0$. There is an instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$, $a < r' < b$. (c) Cylinder of one material for $0 \leqq r < a$ and another material for $a < r < b$. (i) The initial temperature zero, the surface $r = b$ kept at a temperature V_0 for $t > 0$. (ii) The surface $r = b$ kept at zero temperature for $t > 0$. There is an instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$, $0 < r' < a$. (iii) The surface $r = b$ is kept at a zero temperature for $t > 0$. There is an instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$, $a < r' < b$. (iv) The surface $r = b$ is kept at a zero temperature for $t > 0$. The initial temperature is a prescribed function of the radius r. Finally, (d) the region bounded internally by the cylinder $r = a$. From $r = a$ to $r = b$, there is a solid of one material and for $r > b$ of another. (i) The surface $r = a$ maintained at a temperature V_0 for $t > 0$; the initial temperature of the whole body zero. (ii) The surface $r = a$ maintained at zero temperature for $t > 0$. There is an instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$, $a < r' < b$. (iii) The surface $r = a$ maintained at zero temperature for $t > 0$. An instantaneous cylindrical surface of strength Q at $t = 0$ over $r = r'$, $r' > b$. *A. E. Heins* (Cambridge, Mass.).

Schofield, F. H. The steady flow of heat from certain objects buried under flat air-cooled surfaces. Philos. Mag. (7) **31**, 471–497 (1941). [MF 6364]

An approximate method is presented for obtaining steady temperature functions $\theta(x, y)$ in certain types of solids with one plane surface subject to heat transfer according to Newton's law $\partial\theta/\partial n = -h\theta$. The thickness of an additional layer of the material is found such that, when this layer is added at the surface in question and the new boundary is made an isotherm, then the total rate of flow through the original surface will assume its original value. Thus it is a method of added thickness to reduce boundary conditions to the simpler isothermal type, so that the problem may be treated by such a device as the Christoffel-Schwarz transformation. The method is applied to problems of heat transfer from pipes buried in the ground or in a wall, and from girders projecting into a wall. Computations are made which show that the approximate results found here are satisfactory. *R. V. Churchill* (Ann Arbor, Mich.).

Awbery, J. H. Heat flow when the boundary condition is Newton's law. Philos. Mag. (7) **33**, 157–158 (1942). [MF 6259]

One of the problems treated by the author in an earlier paper [Philos. Mag. (7) **7**, 1143–1153 (1929)] was that of steady temperatures in an insulating wall into which a girder projects when there is heat transfer according to Newton's law at one face of the wall. It was pointed out by Schofield [in the paper reviewed above] that the solution given by Awbery does not satisfy the required conditions. It is the purpose of the present note to correct this; but the corrected formula reduces at once to the original because the correction consists only of the representation of an odd function by its Fourier integral in place of the Fourier sine integral originally used. Neither here nor in the original paper is there any consideration of the convergence of the infinite integrals involved, even though some of the integrals are clearly divergent. *R. V. Churchill.*

Gaskell, R. E. A problem in heat conduction and an expansion theorem. Amer. J. Math. **64**, 447–455 (1942). [MF 6444]

Let $U(x, t)$ be the temperature in a uniform cylinder with insulated lateral surface and ends $x = \pm 1$ in contact with fixed quantities of liquids. The initial temperatures of the cylinder, $U(x, 0) = f(x)$, and the liquids are prescribed. It is assumed that the liquids are stirred so that each has a uniform temperature at each instant, and that they may transfer heat to their surroundings according to Newton's law. The problem of determining $U(x, t)$, and hence the temperatures of the liquids, is solved and the solution is rigorously established by means of the Laplace transformation. It is assumed that $f(x)$ is sectionally continuous and $f'(x)$ is bounded and integrable. It follows that any such function $f(x)$ can be represented by a series of the nonorthogonal characteristic functions of the differential system $y'' - \lambda y = 0$, $y'(1) + (a+b)y(1) = 0$, $y'(-1) - (c+\lambda d)y(-1) = 0$, where a, b, c and d are nonnegative constants. Additional requirements on $f(x)$ and $U(x, t)$ are given under which the solution found is the only possible one. *R. V. Churchill.*

Paterson, Stewart. The conduction of heat in a medium generating heat. Philos. Mag. (7) **32**, 384–392 (1941). [MF 6043]

The author derives explicit solutions for a number of problems connected with the heat equation

$$\frac{\partial\theta}{\partial t}-\kappa\Delta\theta=f(x,y,z,t)+g(t)\cdot\theta,\quad \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

His method consists mainly in removing the term $g(t)\cdot\theta$ by the substitution $\theta=v\exp\int_0^t g(\tau)d\tau$. *E. Rothe.*

Avrami, Melvin and Little, J. B. Diffusion of heat through a rectangular bar and the cooling and insulating effect of fins. I. The steady state. J. Appl. Phys. **13**, 255–264 (1942).

Mathematically, the problem reduces to that of finding a function $T(x, y, t)$ satisfying in $0\leqq x\leqq a$, $0\leqq y\leqq b$ (half the cross-section of the bar) the heat equation with the following boundary conditions: $T(x, y, 0)=0$; $T(x, 0, t)=A=\text{const.}$ (contact with a hot plate); $-\mu_1\partial T(x, b, t)/\partial y=T(x, b, t)$ and $-\mu_2\partial T(a, y, t)/\partial x=T(a, y, t)$ (heat exchange with environment of temperature zero); $\partial T(0, y, t)/\partial x=0$ (since $x=0$ is the midsurface of the bar). In the present paper the steady state is studied, that is, the corresponding boundary problem for $\Delta u=0$. An extensive study of T as a function of all the parameters is made. *W. Feller.*

Gatewood, B. E. Thermal stresses in long cylindrical bodies. Philos. Mag. (7) **32**, 282–301 (1941). [MF 5912]

The author reduces the problem of two-dimensional thermal stresses due to an arbitrary temperature T to the determination of a particular integral of the equation $\nabla^2V=kT$, where k is a constant, and to a subsequent determination of an Airy stress function U satisfying the biharmonic function $\nabla^4U=0$. The solution of the latter is carried out along the lines developed by Muschelisvili by the use of two analytic functions of the complex variable $z=x+iy$. Applications are made to a solid cylinder with the temperature a function of the radius, to a hollow cylinder, to several cylinders shrunk one over the other and to the case of two cylinders where the outer boundary of the outer one is eccentric to the common circular boundary of the two. *H. Poritsky* (Schenectady, N. Y.).

Calculus of Variations

Tonelli, Leonida. L'analisi funzionale nel calcolo delle variazioni. Ann. Scuola Norm. Super. Pisa (2) **9**, 289–302 (1940). [MF 5468]

This paper is an exposition of the reasons that led the author to develop his approach to the calculus of variations and of the results obtained by himself and others. He discusses at some length the classical calculus of variations before showing how the functional calculus was brought to bear on the calculus of variations by means of the concept of semi-continuity. Some of the more important results of his work are mentioned in passing, and the essay closes with a few remarks on extensions of the calculus of variations to abstract spaces. *H. H. Goldstine* (Ann Arbor, Mich.).

Cinquini, Silvio. Una nuova estensione dei moderni metodi del calcolo delle variazioni. Ann. Scuola Norm. Super. Pisa (2) **9**, 253–261 (1940). [MF 5465]

The author shows how the methods of Tonelli may be used to secure an existence theorem for a minimum of an integral

$$\int_a^\infty f(x, y, y')dx.$$

At first the region A containing the admissible curves is supposed to be bounded to the left and in the y-direction. Modifications of the usual device allow the bound on y to be removed. The author shows by an example (with $f=y'^2$) that a minimum need not exist under the usual hypotheses when the class K of admissible curves is determined by boundary conditions of the form $y(a)=A$, $y(\infty)=B$. *L. M. Graves* (Chicago, Ill.).

Baiada, Emilio. Sopra un problema di Mayer. Ann. Scuola Norm. Super. Pisa (2) **9**, 109–141 (1940). [MF 5457]

This paper treats a problem of Mayer in parametric form, with two differential equations $u'=F(x, y, x', y', u)$, $v'=G(x, y, x', y', u, v)$, and with the second end-value of u fixed. The quantity to be minimized is the second end-value of v. Thus the problem may be specialized to an ordinary isoperimetric problem. Following some theorems on convergence properties, the author gives four theorems on the existence of a minimum, all rather special in character. In the final section the author shows that under certain circumstances a minimizing curve satisfies differential equations of the form $dK_{x'}/ds=K_x$, $dK_{y'}/ds=K_y$, where K is defined in terms of the functions F and G in a way somewhat more complicated than the usual one. *L. M. Graves* (Chicago, Ill.).

Hestenes, M. R. The problem of Bolza in the calculus of variations. Bull. Amer. Math. Soc. **48**, 57–75 (1942). [MF 6183]

This paper is a survey of the results that have been obtained for the problem of Bolza in the calculus of variations, and a comparative study of the basic ideas and various methods that have been utilized in the treatment of this general problem. The major part of the paper is devoted to the topic of sufficiency proofs for a relative minimum for this problem in nonparametric form. The author concludes with a discussion of the index of an extremal. *W. T. Reid* (Chicago, Ill.).

Bliss, G. A. The calculus of variations for multiple integrals. Amer. Math. Monthly **49**, 77–89 (1942). [MF 6239]

This is a survey of the classical aspect of the theory of multiple-integral problems of the calculus of variations, showing the recent developments and the large remaining gaps. The integral studied is $I=\int_{\mathfrak{X}}f(x, y, y')d\mathfrak{X}$, where $x=(x_1, \cdots, x_m)$, $y=(y_1, \cdots, y_n)$, y' is the matrix of partial derivatives $\partial y_i/\partial x_j$, and $\mathfrak{X}$ is a region in m-space. The minimum of I is sought in a class of functions $y(x)$ assuming assigned values on the boundary $\mathfrak{B}$ of $\mathfrak{X}$. Recent work on the analogues of the Lagrange equations (Haar-Coral form) and ridge condition is discussed, and likewise for the analogues of the Weierstrass, Legendre and Jacobi conditions. There follows a discussion of the concept of a field and of extremals of a field. In particular, when $n=1$ the slope functions of every family of extremals simply covering a region is a field, and for $(m, n)=(2, 1)$ a reasonably adequate sufficiency theorem is known. Even if m and n both exceed 1 the general form of the invariant integral is known. For this case some sufficiency theorems can be stated, but their hypotheses do not match the necessary conditions, and the theory is far from complete. *E. J. McShane.*

Young, L. C. Generalized surfaces in the calculus of variations. Ann. of Math. (2) **43**, 84–103 (1942). [MF 6063]

With this note the author, who devised generalized curves, begins a study of the analogously defined generalized surfaces (in nonparametric form). A generalized surface is a system S^* consisting of (i) a function $z(x, y)$ absolutely continuous in the sense of Tonelli, (ii) for each (x, y) a linear functional M which assigns to each continuous function $g(p, q)$ an "average value" which is measurable as a function of (x, y) and is such that the means of p, q are almost everywhere z_x, z_y, respectively. S^* is Lipschitzian if for some K the average $M(g)$ depends only on the values of $g(p, q)$ within $p^2+q^2 \leqq K^2$. If $f(x, y, z, p, q)$ is continuous, its integral $F(S^*)$ over S^* is obtained by substituting $z(x, y)$ for z, applying M to obtain a function $\phi(x, y)$, and integrating this. A sequence S_n^* converges (to a limit S_0^*) if for every integrand f the sequence of numbers $F(S_n^*)$ converges (to $F(S_0^*)$). It is shown that every convergent sequence converges to a limit, and that a family of generalized surfaces is conditionally compact if and only if their tracks $z=z(x, y)$ lie in a sphere and they are uniformly Lipschitzian. Much deeper than its analogue for curves is the theorem that in a class of generalized surfaces all of which satisfy the same Lipschitz condition and have the same boundary, the subclass of ordinary surfaces is dense. It follows readily that, if we seek to minimize an integral in the class of all surfaces having assigned boundary values and Lipschitz constant, the lower bound is the same whether we restrict attention to ordinary surfaces or allow generalized surfaces, and on the latter class the minimum is altered. *E. J. McShane* (Charlottesville, Va.).

Giuliano, Landolino. Sulle condizioni sufficienti per la semicontinuità degli integrali doppi del calcolo delle variazioni. Ann. Scuola Norm. Super. Pisa (2) **10**, 37–55 (1941). [MF 5471]

Tonelli showed [Acta Math. **53** (1929)] that the integral

$$I(z)=\int\int_D F(x, y, z, p, q)dxdy$$

is lower semi-continuous on the class of functions $z(x, y)$ which are absolutely continuous in the sense of Tonelli on the open set D provided that the following hypotheses are satisfied. (i) The integral $I[z]$ is positive quasi-regular. (ii) The derivatives F_p, F_q, F_{px}, F_{qy} exist and are continuous. (iii) Either (α) the integrand F is bounded below or (β) the boundary of D is a rectifiable Jordan curve. In the present note it is shown that (ii) can be weakened to read "F is continuous" provided that (i) is strengthened to read "The integral $I[z]$ is positive quasi-regular seminormal." By this it is meant that for fixed x, y, z the function $u=F(x, y, z, p, q)$ is convex and has a plane of support which touches it at only a single point. As a corollary, if K is a class of surfaces $z=z(x, y)$ absolutely continuous in the sense of Tonelli on D and of uniformly bounded lengths, and F is continuous, and (i) and (iii) hold, then $I[z]$ is lower semi-continuous on K. The proofs are essentially different from those of Tonelli cited above, and rest on the following lemma. If $I[z]$ is positive quasi-regular semi-normal, and (x_0, y_0) is in D, and $\epsilon>0$, $z_0, \bar{p}, \bar{q}$ are arbitrary, there are numbers $\rho>0$, λ, μ, ν such that if (x, y, z) has distance less than ρ from (x_0, y_0, z_0) the inequality $f(x, y, z, p, q)>\lambda p+\mu q+\nu$ holds for all p, q and the inequality $f(x, y, z, p, q)<\lambda p+\mu q+\nu+\epsilon$ holds if (p, q) has distance less than ρ from $(\bar{p}, \bar{q})$. *E. J. McShane.*

Amerio, Luigi. Studi su gli integrali doppi del calcolo delle variazioni. Ann. Scuola Norm. Super. Pisa (2) **10**, 57–89 (1941). [MF 5472]

This paper is concerned with some modifications of Tonelli's theorems on semi-continuity, stated in the preceding review, and also with some existence theorems. If M is a positive number, and F exceeds a function $f(x, y)$ summable over D whenever $|p| \leqq M$ and $|q| \leqq M$, and $I[z]$ is positive quasi-regular, then $I[z]$ is lower semi-continuous on the class of surfaces satisfying a Lipschitz condition of constant M. [This result, however, is readily deducible from known theorems.] A lemma on extension of range of functions is established. [A more precise result is already known; cf. E. J. McShane, Bull. Amer. Math. Soc. **40**, 837–842 (1934), in particular, p. 838.] This leads to a theorem on the existence of a minimizing surface in a class of uniformly Lipschitzian surfaces; this too is readily deducible from known results. Another semi-continuity theorem relaxes hypothesis (iiiβ) [cf. review above] by supposing that D can be approximated from within by a swelling sequence of regions each bounded by a finite number of Jordan curves of uniformly bounded total lengths; F is assumed bounded below by a linear function of p and q which satisfies certain requirements. An existence theorem is established for the minimum of an integral $I[z]$ in which F exceeds a function of the form $\mu\{|p|^{2+\alpha}+|q|^{2+\alpha}\}+M$, $\alpha>0$, except near a point or smooth curve in (x, y, z)-space, in a neighborhood of which it exceeds $\nu\{|p|^{1+\beta}+|q|^{1+\beta}\}+N$, $\beta>0$. *E. J. McShane* (Charlottesville, Va.).

Giuliano, Landolino. Osservazioni sopra alcuni teoremi di semicontinuità degli integrali doppi. Ann. Scuola Norm. Super. Pisa (2) **10**, 115–122 (1941). [MF 6417]

Tonelli has shown that, if D is a bounded open set in the (x, y) plane, and $f(x, y, z, p, q)$ possesses certain partial derivatives and has a nonnegative E-function, the integral of f over D is a lower semicontinuous functional provided that either of the following hypotheses holds. (A) f is bounded below. (B) D is bounded by a rectifiable simple closed curve, and f and certain of its partial derivatives are continuous for all (x, y) on D plus boundary. Here it is shown that (A) can be replaced by (A′): There are functions $\varphi_1(x, y)$, $\varphi_2(x, y)$, both summable and φ_2 bounded, such that $f(x, y, z, p, q) \geqq \varphi_1(x, y)+z\varphi_2(x, y)$. Also (B) can be replaced by (B′): The boundary of D consists of a countable collection of simple closed curves of finite total length, and f and certain partials are continuous for (x, y) on D plus boundary. It is also possible to replace (B) by (B″): There are functions $\varphi_1(x, y), \cdots, \varphi_4(x, y)$ such that φ_2 and φ_4 are bounded and continuous on D, φ_3 bounded and summable on D, φ_1, $\partial\varphi_2/\partial x$, $\partial\varphi_4/\partial y$ summable on D, with which the inequality $f(x, y, z, p, q) \geqq \varphi_1(x, y)+z\varphi_2+p\varphi_3+q\varphi_4$ holds. This last theorem generalizes a result of Amerio [see the preceding review]. *E. J. McShane.*

Morse, Marston and Tompkins, C. The continuity of the area of harmonic surfaces as a function of the boundary representations. Amer. J. Math. **63**, 825–838 (1941). [MF 5628]

Let M be the aggregate of m-tuples of functions $p_i(\vartheta)$ $(i=1, \cdots, m;\ 0 \leqq \vartheta \leqq 2)$ defined and of bounded variation on a circumference $r=$const. The distance between two m-tuples p_i, q_i of M is defined by the function

$$(*) \qquad pq=\sum_i\{|V_i(p)-V_i(q)|+\max|p_i(\vartheta)-q_i(\vartheta)|\},$$

where $V_i(p)$ is the total variation of p_i on $0 \leqq \vartheta \leqq 2$; this satisfies the axioms for a metric. Let $\Omega(p)$ be the area of the harmonic surface defined by the boundary-values $p_i(\vartheta)$. The principal theorem of this paper states that $\Omega(p)$ is a continuous function on M. This is not true of Douglas' functional, nor can the terms $|V_i(p)-V_i(q)|$ be omitted from the definition of distance. More generally, let B be a region on a Riemann surface whose boundaries are non-intersecting circles $C_1, \cdots, C_k$, the center of C_j being (u_j, v_j) and its radius being r_j. On C_j let $p_{j,1}(\vartheta), \cdots, p_{j,m}(\vartheta)$ be functions of bounded variation, and let

$$\Omega(p_1, \cdots, p_k, u_1, v_1, r_1, \cdots, u_k, v_k, r_k)$$

be the area of the harmonic surface defined on B and having boundary values on C_j given by $p_{j,i}(\vartheta)$. The area Ω is a continuous function of all its arguments, if for the arguments p_j we use the metric (*) of the space M.

E. J. McShane (Charlottesville, Va.).

Morse, Marston. A mathematical theory of equilibrium with applications to minimal surface theory. Science **93**, 69–71 (1941). [MF 5971]

Shiffman, Max. Unstable minimal surfaces with any rectifiable boundary. Proc. Nat. Acad. Sci. U. S. A. **28**, 103–108 (1942). [MF 6332]

The writer proves the following principal theorem: If the rectifiable closed Jordan curve Γ bounds two minimal surfaces which are proper relative minima, then Γ must bound an unstable minimal surface. The lemma which enables the writer to handle the case of rectifiable boundaries is the following: If a sequence $\{S_n\}$ of minimal surfaces (of disc type) in conformal representation converges uniformly to a minimal surface S and if the lengths of the boundaries of the S_n tend to that of S (all finite), then the areas of the S_n tend to that of S. The writer considers the space $\mathfrak{P}$ of potential vectors defined on the unit circle Σ which map its boundary C in a monotonic way on the given curve Γ and satisfy a preassigned three point condition. The writer then obtains a sequence $K^{(\nu)}$ of extensions of $\mathfrak{P}$ such that $K^{(1)} \supset K^{(2)} \supset \cdots \to \mathfrak{P}$, each $K^{(\nu)}$ containing potential functions mapping C on a certain class of curves near Γ. Then, if Γ bounds two minimal surfaces which are proper relative minima, the writer shows the existence of unstable minimal surfaces $S^{(\nu)}$ in $K^{(\nu)}$, a subsequence of which converges uniformly (in conformal representation) to a minimal surface S in $\mathfrak{P}$ which is then shown to be unstable.

C. B. Morrey, Jr. (Berkeley, Calif.).

Shiffman, Max. Unstable minimal surfaces with several boundaries. Ann. of Math. (2) **43**, 197–222 (1942). [MF 6459]

The writer extends the theory of unstable minimal surfaces bounded by one contour to the case of surfaces of genus zero bounded by k nonintersecting Jordan contours $\Gamma_1, \cdots, \Gamma_k$. Let distinct points P_μ, Q_μ and R_μ be chosen on each Γ_μ. The writer begins by considering a space $\mathfrak{R}_k$, each element G of which consists either of an ordinary region bounded by k circles and distinct points p_μ, q_μ, r_μ on each boundary C_μ or a "degenerate limiting element" which consists of several such regions, each bounded by some of the boundary circles C_μ with points p_μ, q_μ and r_μ on each, and certain other "point circles" c_μ, c_μ', etc., each of which is identified with points (similarly indicated) on some of the C_μ; all the regions and points are normalized in a certain way. An element of the space $\mathfrak{P}$ consists of a potential vector $\zeta(u, v)$ defined on the closed regions of an element G of $\mathfrak{R}_k$ such that ζ maps the boundary circle C_μ monotonically onto Γ_μ carrying p_μ, q_μ and r_μ into P_μ, Q_μ and R_μ and ζ is constant on each point circle c_μ of C_μ. A metric is introduced into both $\mathfrak{R}_k$ and $\mathfrak{P}$ and it is shown that the subset $\mathfrak{P}_N$ on $\mathfrak{P}$ for which $D(\zeta) \leqq N$ is compact and closed for every N. The principal theorems of the writer are as follows: (A) Suppose that the reducibility condition holds for each Γ_μ individually. Then the Morse theory applies to the minimal surfaces (degenerate as well as ordinary) bounded by $\Gamma_1, \cdots, \Gamma_k$. (B) If the Γ_μ are polygons, the Morse theory applies as above and the connectivity numbers of $\mathfrak{P}$ are: $R_0=1$, $R_1=R_2=\cdots=0$. In particular, if M_n is the sum of the nth type numbers of all blocs of minimal surfaces bounded by $\Gamma_1, \cdots, \Gamma_k$, and if each M_n is finite, then $M_n - M_{n-1} + \cdots + (-1)^n M_0 \geqq (-1)^n$ for each n and $M_n=0$ for all $n > 3k-6+V$, where V is the total number of vertices in all the Γ_μ. *C. B. Morrey, Jr.*

Davids, Norman. Minimal surfaces spanning closed manifolds and having prescribed topological position. Amer. J. Math. **64**, 348–362 (1942). [MF 6437]

The writer considers a closed oriented manifold M of first Betti number $p>0$ situated in n-dimensional Euclidean space E_n. Let $h_1, \cdots, h_p$ be any one-dimensional homology basis for M; then there exist p oriented finite polyhedral cycles $z_1, \cdots, z_p$ in $E_n - M$ and at a distance $\alpha > 0$ from M such that $\lambda(h_i, z_j) = \delta_{ij}$, λ denoting the linking number. Any homology class on M is equivalent to its set of linking numbers. The writer then considers vector surfaces ζ each of which consists of a finite number of surfaces $\zeta_r (= \zeta_r(u, v))$, each defined and continuous on the interior of a finitely connected Jordan region G_r. Such a vector surface is said to span M if, for each sequence $\{(u_n, v_n)\}$ in any G_r which tends to the boundary G_r^* of G_r, the distance of $\zeta_r(u_n, v_n)$ from M tends to zero. Since the ζ_r are not assumed continuous on G_r^*, the homology class of the boundary of ζ is defined as follows: In each G_r let there be chosen sequences of simple closed curves, each such sequence tending to a boundary curve of G_r; for n sufficiently large the distance from M of the images of these curves under the ζ_r will all be less than $\alpha/2$, so that these curves will all have the same linking numbers with respect to the z_i; the homology class of the boundary of ζ is defined as that defined by these linking numbers. The admissible vectors ζ are defined by the conditions: (a) each ζ_r is of class D' on the interior of G_r; (b) the Dirichlet integral of ζ is bounded; (c) ζ spans M; (d) the boundary of ζ belongs to the prescribed homology class on M; (e) each surface of ζ is orientable and the total characteristic $\rho(\zeta)$ is less than a fixed constant K. The writer then proves: Given a homology class on M defined by $\lambda(B, z_i) = \lambda_i$ (B denoting the total boundary of ζ), the λ_i not being all zero, for which the class of admissible vectors is not empty, there exists a vector ζ in this class for which the Dirichlet integral attains its minimum value $\delta(\lambda_1, \cdots, \lambda_p) > 0$; each surface ζ_r of ζ is a minimal surface and $\rho(\zeta) < K$. Higher topological types are considered, using identifications between the boundaries of the G_r. A sufficient condition that the minimizing ζ consists of one surface is that $\delta(\lambda_1, \cdots, \lambda_p) \leqq \delta(\lambda_1', \cdots, \lambda_p')$ for any set $(\lambda_1', \cdots, \lambda_p')$. *C. B. Morrey, Jr.* (Berkeley, Calif.).

GEOMETRY

Fraile, V., Fraile, A. and Crespo, C. The locus and the loci of point domains in the plane. Revista Union Mat. Argentina **7**, 87–91, 114–119, 144–169 (1941). (Spanish) [MF 6182]

Continuation and conclusion of the paper, the first part of which appeared in the same Revista **7**, 45–50 (1941); cf. these Rev. **2**, 294.

Mehr, Emanuel. The geometry of the triangle in the Kasner plane. Amer. Math. Monthly **48**, 535–539 (1941). [MF 5717]

E. Kasner, in his studies on horned angles, has introduced a geometry of the plane in which the distance of two points (x_1, y_1), (x_2, y_2) is given by $d_{12}=(x_2-x_1)^2/(y_2-y_1)$ and the angle of two lines $y=p_1x+q_1$ and $y=p_2x+q_2$ is $\theta_{12}=p_2/p_1$. Rigid motions are given by the group G_3: $x'=mx+h$, $y'=m^2y+k$, and similitude transformations by the G_4: $x'=mx+h$, $y'=ny+k$. There is no shortest distance from a point to a line. Perpendicularity is defined by $\theta_{12}=\frac{1}{2}$, $\theta_{21}=2$, and "quasi-perpendicularity" by $\theta_{12}=\theta_{21}$. This last definition is used for altitudes of a triangle, and for bisectors of the sides of a triangle. The bisector of the angle θ_{12} is defined by the equation $\theta_{13}=\theta_{23}$. A number of theorems of the ordinary triangle are now generalized for this case, so that it is possible to define the orthocenter, the circumcircle, the nine-point circle, the line of Euler and the median point. *D. J. Struik* (Cambridge, Mass.).

Patterson, B. C. The inversive plane. Amer. Math. Monthly **48**, 589–599 (1941). [MF 5894]

This expository paper compares and contrasts two methods of studying inversive geometry. The first method is that of Kasner which uses isotropic coördinates for complex points. The second method is that of Morley, which uses circular coördinates for pairs of real points.

P. Franklin (Cambridge, Mass.).

Weiss, E. A. Konstruktionen mit hängenden Linealen. Deutsche Math. **6**, 3–15 (1941). [MF 5802]

The paper discusses some of the fundamental constructions of elementary projective geometry.

G. de B. Robinson (Toronto, Ont.).

Peters, J. W. The theorem of Morley. Nat. Math. Mag. **16**, 119–126 (1941). [MF 5831]

Gibbins, N. M. The non-equilateral Morley triangles. Math. Gaz. **26**, 81–86 (1942). [MF 6667]

Gentry, F. C. Analytic geometry of the triangle. Nat. Math. Mag. **16**, 127–140 (1941). [MF 5832]

Wood, P. W. Points isogonally conjugate with respect to a triangle. Math. Gaz. **25**, 266–272 (1941). [MF 6269]

Allen, E. F. On a triangle inscribed in a rectangular hyperbola. Amer. Math. Monthly **48**, 675–681 (1941). [MF 6027]

Ramesam, V. A further note on Feuerbach's theorem. Math. Student **9**, 129–130 (1941). [MF 6476]

Saddler, W. Polar properties and canonical forms. J. London Math. Soc. **16**, 167–172 (1941). [MF 6013]

The theorem that two triangles polar reciprocal with respect to a conic are perspective [Michel Chasles, Gergonne's Ann. Math. Pures Appl. **19**, 74–75 (1828)] and its extension to tetrahedrons [M. Chasles, ibid, p. 76] may, according to the author, be further extended to space of n dimensions by a method which he uses to prove the proposition for $n=4$.

N. A. Court (Norman, Okla.).

Gulasekharam, F. H. V. The orthopolar circle. Math. Gaz. **25**, 288–297 (1941). [MF 6271]

Goormaghtigh, R. A study of a quadrilateral inscribed in a circle. Amer. Math. Monthly **49**, 174–181 (1942). [MF 6394]

Rangachariar, V. On some properties of rectangular hyperboloids. Math. Student **9**, 131–133 (1941). [MF 6477]

Følner, Erling. The type of a conic section which is determined by five points or five lines. Mat. Tidsskr. B. 1941, 42–51 (1941). (Danish) [MF 6174]

Patterson, B. C. Jacobian circles of the biquadratic. Amer. Math. Monthly **49**, 304–309 (1942). [MF 6693]

Weaver, J. H. Properties of points, lines, and circles associated with a point on an ellipse. Amer. Math. Monthly **48**, 435–443 (1941). [MF 5536]

Weaver, J. H. On the cubic of Tschirnhausen. Nat. Math. Mag. **16**, 371–374 (1942). [MF 6686]

Rossier, Paul. Sur les courbes anallagmatiques et circulaires. C. R. Soc. Phys. Genève **58**, 82–84 (1941). [MF 6491]

Rossier, Paul. Sur les cercles osculateurs aux courbes anallagmatiques et circulaires. C. R. Soc. Phys. Genève **58**, 168–170 (1941). [MF 6500]

Thébault, V. The Adams sphere. Amer. Math. Monthly **49**, 170–173 (1942). [MF 6393]

Neville, E. H. The isoptic point of a quadrangle. J. London Math. Soc. **16**, 173–174 (1941). [MF 6014]

The proposition that, given four points in the plane, there exists a point whose powers for their four triad-circles are proportional to the squares of the radii of the circles, is proved by an analytical method "which shows clearly why no extension to space of three or more dimensions is to be expected." *N. A. Court* (Norman, Okla.).

Court, N. A. On the anharmonic associates of a point for a triangle and a tetrahedron. Bol. Mat. **15**, 14–16 (1942). [MF 6603]

Hope-Jones, W. The regular octohedron. Math. Gaz. **26**, 41–46 (1942). [MF 6326]

Macmillan, R. H. An account of 4-piece mechanisms in three dimensions. Math. Gaz. **26**, 5–20 (1942). [MF 6325]

Two special four link mechanisms, the "spheric" chain and the "skew isogram," are investigated. They are studied

both by a direct use of orthogonal projections, and by means of spherical trigonometry. Some practical applications are suggested. *P. Franklin* (Cambridge, Mass.).

Goldberg, Michael. Polyhedral linkages. Nat. Math. Mag. **16**, 323–332 (1942). [MF 6482]

A clear account of certain space linkages, which consist of rigid flat plates hinged together so as to have one degree of freedom when one of the plates is fixed. Among these is a six-plate linkage, discovered by P. F. Sarrus in 1853, which traces straight lines in space (and so excels Peaucellier's more celebrated linkage, which merely traces straight lines in a given plane). *H. S. M. Coxeter.*

Benedikt, E. T. Sopra una proprietà delle omografie. Bol. Mat. **14**, 291–294 (1941). [MF 5893]

The author investigates the quadric in three-space which is transformed, by a given linear vector transformation, into the unit sphere, and he examines the equality of the degree of degeneration of this surface and of the pure strain connected with the given transformation. *E. Helly.*

Chatterji, N. and Dasgupta, P. N. On some congruence quadrics obtained from linear complexes of the irreducible system of two quaternary quadrics with two linear complexes. Bull. Calcutta Math. Soc. **32**, 43–50 (1940). [MF 6155]

The invariants and covariants of the system of two quaternary quadrics and two linear complexes were discussed by the authors in an earlier paper [Bull. Calcutta Math. Soc. **27**, 183–200 (1935)]. In this paper, a few of the associated quadrics of the system are discussed, and a complete list of the irreducible linear complexes of the system is given, in symbolic form. *E. Helly.*

Ciani, E. Sopra un gruppo notevole di collineazioni piane. Boll. Un. Mat. Ital. (2) **3**, 177–187 (1941). [MF 5583]

Les 6 points singuliers sur un plan singulier d'une configuration de Kummer plusieurs fois tetraédrale appartiennent à une conique, sur laquelle ils forment trois couples en involution. Un même exagone peut appartenir de cette manière à une, deux, trois, quatre ou six involutions. En revenant sur une étude déjà ancienne, développée en 1897 par les méthodes de la géométrie synthétique, l'auteur étudie ici par la représentation analytique le cas de l'exagone dont les sommets se disposent de quatre façons en des couples en involution (tetraexagone selon l'auteur), duquel il donne un modèle réel sur la conique $x^2-y^2+z^2+xz=0$. En plus de ces quatre involutions, existent trois autres qui laissent invariant l'exagone, deux sommets étant les points doubles. L'auteur décrit le groupe de collineations qui contient toutes ces involutions et certain nombre de propriétés de systèmes de coniques et de cubiques liés à l'exagone. On peut observer qu'un modèle tout simple de la configuration étudiée est fournit par l'exagone régulier. *B. Levi.*

Schönberg, Mario. On an extension of the spinor calculus. Anais Acad. Brasil. Sci. **13**, 129–144 (1941). (Portuguese) [MF 6584]

Projective coordinates are introduced on the absolute of a non-Euclidean space. The projectivities of the non-Euclidean space which leave the absolute invariant induce "spin" transformations on these projective coordinates for the absolute. The conformal group of a Euclidean space is isomorphic to the group of projectivities leaving the absolute invariant in a suitably chosen non-Euclidean space. Hence one can also discuss a spin representation for that conformal group. *A. Schwartz* (State College, Pa.).

***Roever, William H. Fundamental Theorems of Orthographic Axonometry and Their Value in Picturization.** Washington University Studies, New Series, Science and Technology, no. 12, 1941. v+47 pp. $1.00.

Continuation of the author's book on "The Mongean Method of Descriptive Geometry" [Macmillan, New York, 1933] dealing with orthographic axonometry.

O. Neugebauer (Providence, R. I.).

Rossier, Paul. Sur une règle pratique de dessin géométrique. C. R. Soc. Phys. Genève **58**, 84–86 (1941). [MF 6492]

Graf, Ulrich. Über das Photo eines Photos. Z. Angew. Math. Mech. **21**, 183–189 (1941).

Suppose F is the photograph (that is, perspective representation) of an object O and F' a photograph of F. Obviously F' is in general not again a photograph of O. The author shows that, outside of the trivial case $F'\equiv F$, there exists only one direction from which F' is again a photograph F'' of O, and F'' corresponds to a picture of O taken from the same point as F but in a direction which depends on F. *O. Neugebauer* (Providence, R. I.).

Dilloway, A. J. The cartographic solution of great circle problems. J. Royal Aeronaut. Soc. **46**, 4–31 (1942). [MF 6374]

The recent development of long-distant air travel asks for solutions of great circle problems. A few of the fundamental problems to be solved are: Given the geographical coordinates of two points, determine the great circle between them, and the initial bearing of one from the other; given the geographical coordinates of one point, determine the locus of a second point when the distance between them (that is, zenith distance) and a bearing of the known distance, measured at an unfixed position, are given. The traditional solutions of such problems by spherical trigonometry is not as rapid as required by air navigation. This paper discusses the cartographic solution of great circle problems; it examines graphical and mechanical methods which can be used in connection with a map on which the positions of the points are plotted. The author gives a detailed record of the different methods and mechanical devices which, up to the present, have been developed with respect to certain map projections. A few of the author's own investigations are included, and the relative value and the precision of the different methods is discussed.

E. Helly (Long Branch, N. J.).

Anadon Laplaza, Santos. The conformal cartography of the ellipsoid of revolution with applications to the map of Spain. Revista Mat. Hisp.-Amer. (3) **2**, 8–11 (1940). (Spanish) [MF 5046]

Dufton, A. F. The world mapped in complementary segments. Philos. Mag. (7) **32**, 436 (1941). (1 plate) [MF 6045]

An equi-areal projection is given on two complementary segments of the sphere. These segments are limited by the intersections of the sphere $x^2+y^2+z^2=r^2$ with $2z^2=r^2$ for $x\geqq 0$ and $2y^2=r^2$ for $x\leqq 0$, respectively. *O. Neugebauer.*

Convex Domains, Integral Geometry

Perron, Oskar. Modulartige lückenlose Ausfüllung des R_n mit kongruenten Würfeln. II. Math. Ann. 117, 609–658 (1941). [MF 5649]

In part I [Math. Ann. 117, 415–447 (1940); these Rev. 2, 153] the author established Minkowski's conjecture for Euclidean space of $n \leqq 8$ dimensions. The present part consists of the extension to nine dimensions. No fresh principles are involved; but the multitude of cases and sub-cases renders the work formidable indeed, and seems to indicate that the extension to still higher values of n would not be humanly feasible by this method. *H. S. M. Coxeter.*

Hadwiger, H. Über Parallelinvarianten bei Eibereichen. Comment. Math. Helv. 13, 252–256 (1941).

If G_1 and G_2 are two plane ovals with areas F_1 and F_2 and circumferences U_1 and U_2, respectively, the author defines a parallel-invariant $\Phi(F_1, F_2, U_1, U_2)$ for G_1 and G_2 as a function which is unaltered if G_1 and G_2 are replaced by their outer parallel ovals $G_1(\xi)$ and $G_2(\xi)$ at distance ξ. Such an invariant is shown to satisfy

$$\sum_{i=1}^{2}\left(U_i\frac{\partial\Phi}{\partial F_i}+2\pi\frac{\partial\Phi}{\partial U_i}\right)=0.$$

This partial differential equation has the three solutions $\Phi_{ik}=F_i+F_k-U_iU_k/2\pi$, $i, k=1, 2$, and any solution is of the form $\varphi(\Phi_{11}, \Phi_{12}, \Phi_{22})$. The remainder of the paper consists in obtaining geometric interpretations of the fundamental invariants Φ_{ik}. One of these is as follows: Let G_k^θ be the oval G_k rotated through the angle θ, and $G_i-G_k^\theta$ be the difference region of the two ovals in the sense of Minkowski linear combinations. Then Φ_{ik} is equal to the average extended over all rotations $0 \leqq \theta < 2\pi$ of the area of the difference region $G_i-G_k^\theta$. A number of other geometric interpretations are also obtained. *J. W. Green.*

Santaló, L. A. Sur quelques problèmes de probabilités géométriques. Tôhoku Math. J. 47, 159–171 (1940). [MF 4016]

The author computes the probabilities that a segment of fixed length intersects certain special uniform nets in a given number of points [cf., for example, L. A. Santaló, Publ. Inst. Math. Univ. Nac. Litoral 1, no. 2 (1939); these Rev. 1, 265]. *P. Scherk* (Bloomington, Ind.).

Raimondi, Elba. On the pairs of secants of a polygon. Revista Union Mat. Argentina 7, 133–134 (1941). (Spanish) [MF 6356]

The author proves that the probability that two straight lines which intersect a regular polygon of $2n$ sides form an angle less than $\pi/2n$ with each other is equal to $1/n$. *P. Scherk* (Bloomington, Ind.).

Raimondi, Elba R. On a problem of geometrical probabilities. Revista Union Mat. Argentina 7, 106–109 (1941). (Spanish) [MF 6181]

Santaló, L. A. Generalization of a problem of geometrical probabilities. Revista Union Mat. Argentina 7, 129–132 (1941). (Spanish) [MF 6355]

Given any two pairs of points on the perimeter of a convex polygon C [Raimondi] or of an arbitrary convex domain C [Santaló]. Each of the pairs determines a straight line. The authors compute the probability that the two straight lines intersect in the interior of C. *P. Scherk.*

Gaspar, Eduardo. Integral formulas concerning the intersection of a plane figure with variable strips. Publ. Inst. Mat. Univ. Nac. Litoral 2, 115–138 (1940). (Spanish) [MF 3708]

A strip B in the Euclidean plane is bounded by two parallel straight lines. Let p be the distance of the middle line of B from a fixed point, and let φ be its angle with a fixed direction. The width $\Delta=\Delta(\varphi)$ of B shall depend upon φ only; $\Lambda=\int_0^\pi\Delta(\varphi)d\varphi$. The density of strips is defined through $dB=dpd\varphi$. Notations: K=domain with a sufficiently smooth perimeter of length L and of total curvature $C(=2\pi)$; F=area of K; f, l, c denote the sums of the corresponding quantities for each part of the intersection BK. The author proves various integral formulas, for example, $\int fdB=F\Lambda$ (proved for convex K only), $\int ldB=2\pi F+L\Lambda$, $\int cdB=2\pi L+\Lambda C$ (especially $\int_{BK\neq 0}dB=L+\Lambda$ for convex K), and generalizes them for intersections of K with n strips, etc.; some of his formulas hold under weaker assumptions. *P. Scherk* (Bloomington, Ind.).

Chern, Shiing-shen. On integral geometry in Klein spaces. Ann. of Math. (2) 43, 178–189 (1942). [MF 6069]

This paper constitutes a praiseworthy attempt to bring methods and results of the so-called "integral geometry" where they really belong, that is, within the framework of E. Cartan's theory of homogeneous spaces. Those (called Klein spaces by the author) are the spaces in which a transitive Lie group G of automorphisms has been defined, and which can therefore be represented as spaces of cosets G/g in G with respect to a subgroup g. As the author has realized, integral geometry is the study of a certain type of relations between integral invariants in two homogeneous spaces $N=G/g$, $M=G/h$ defined by the same group G and different subgroups g, h, and is by no means confined to the group of motions in a Euclidean space (or in one of constant curvature). The best-known case, and the one which may best serve as illustration, is of course that in which G is the group of motions in Euclidean space E^n, where $n=2$ or 3, and g, h are the subgroups which respectively leave invariant a given straight line and a given point; G/h is then E^n itself (viewed as a point-space), and G/g the space of straight lines in E^n.

§ 1 sketches in brief outline Cartan's method of invariant differential forms as applied to homogeneous spaces; it gives, in terms of the constants of structure, a necessary and sufficient condition for the existence of an invariant measure, expressible by the integral of an invariant differential form, in a homogeneous space [cf. the treatment of the same problem for arbitrary locally compact groups in the recent monograph by the reviewer, "L'intégration dans les groupes topologiques," Actual. Sci. Ind., no. 869, Hermann et Cie., Paris, 1941; these Rev. 3, 198]. The condition, as given, is purely local; the further considerations which are necessary if orientation is taken into account (g need not be connected, nor need G/g be orientable) are omitted.

§ 2 proposes a general definition of "incidence" between an element a in $M=G/h$ and an element b in $N=G/g$; a, b are called "incident" if the cosets $x\cdot h$, $y\cdot g$ which define them have a point z in common, in which case they of course have in common the whole coset $z\cdot\gamma$ with respect to $\gamma=h\cap g$. This amounts to saying that there is a transformation $u=x^{-1}$ in the group which brings a, b to a given canonical position $u(a)=a_0$, $u(b)=b_0$ (corresponding to the particular cosets h, g). This clearly includes the usual defi-

nition of "incidence" in all relevant particular cases, but is much more general. In the case of points and straight lines in E^n, for instance, it will mean that a point and straight line are called "incident" in the sense of the author if they are at a given distance δ from each other, δ being the distance between a_0 and b_0; this is the same as the usual "incidence" if $\delta=0$; still calling it "incidence" for $\delta>0$ may be slightly misleading, but only serves to widen the scope of the author's results.

§ 3 is an attempt to prove the analogue of Crofton's and Cauchy's formulae in the general setting previously outlined; the generalized Crofton formula alone is discussed in full detail. Crofton's classical theorem asserts that the measure of the set of straight lines in E^2 which are incident with a given curve is, up to a universal constant factor, the length of the curve; the same is true in E^3, curves and their lengths being replaced by surfaces and their areas. The author now considers the two homogeneous spaces $M=G/h$, $N=G/g$, of dimensions m, n, respectively; let V_p be a portion of a p-dimensional variety in M, the dimension p being such that the set V_n of elements of N which are incident with some elements of V_p is of dimension n. The main part of the author's argument then consists in finding a suitable expression of the differential form which defines the invariant measure in N, when that form is to be calculated over V_n. To that effect, a system of representatives W_p is chosen in G for the cosets which correspond to the elements of V_p; the calculation then proceeds correctly, although notations are somewhat confusing, and the advantages of Cartan's methods are partly lost by the unnecessary introduction of local coordinates and the implicit use of systems of representatives; the assertions [p. 185] that the forms $\bar{\omega}^i$ are "Pfaffian forms in λ^r only," and that $a_k{}^i$ "are functions of λ^r," would be trivial if systems of representatives had been chosen in h for the cosets of γ, but are untrue otherwise. Most of the calculation is valid for an arbitrary choice of the system of representatives W_p for V_p in G; the author, however, assumes from the beginning that this choice has been made in a definite manner "by the method of moving frames of Cartan," and uses this in trying to define the invariant p-dimensional measure (or "area") of V_p in M; unfortunately, the treatment is exceedingly sketchy at this point, and the phrase "it is in general possible" [p. 182] hardly helps to dissipate the obscurities of the proposed definition. There is even more vagueness in the carrying out of the integration of the calculated differential form; the author only says that he does this "by first holding an element of V_p fixed"; doing so without any precaution, however, would give 0 even in such a simple case as that of the classical Crofton formula; in the latter case, what is done is to integrate, not the form itself, but its absolute value; and it is open to question whether the same device should be adopted in the general case, or whether (as the author seems to suggest in the case of infinite integrals) integration should be carried out over some part only of the set V_n. It is apparent, from the work of G. de Rham on integral invariants [Jber. Deutsch. Math. Verein. 49, 156–161 (1939); these Rev. 1, 119] that the invariant p-dimensional measure in a homogeneous space M is not always uniquely defined, and therefore deeper investigations probably would be required in order to settle those questions [cf. also W. Maak's work on Crofton's formula, Math. Ann. 116, 574–597 (1939)]. It would seem, therefore, that the author has not conclusively proved any general theorem on the subject; however, he has given a method which is probably adequate to settle any particular case where we already have some information about the application of the moving frame method; and it should again be emphasized that the main calculation is correct and should be found useful in any further work on the subject. Similar remarks apply to the author's attempt at a generalization of Cauchy's formula, which is given only in brief outline, and depends upon the same calculation as above indicated. *A. Weil* (Haverford, Pa.).

Chern, Shiing-shen. Sur une généralisation d'une formule de Crofton. C. R. Acad. Sci. Paris 210, 757–758 (1940). [MF 4888]

The author announces without proofs a generalization to a Klein space of a formula of Crofton on geometrical probabilities in the Euclidean plane. Cf. the preceding review, § 3.

MATHEMATICAL PHYSICS

Gause, G. F. Note on a possible application of some concepts of topology to asymmetric organization of protoplasm. Bull. Math. Biophys. 3, 127–128 (1941). [MF 5891]

Since the various substances participating in the fundamental metabolic processes of living matter possess the same (left) steric configuration the author says that these molecules form an equioriented manifold. He then applies the "forbidden" operation of inversion on a Möbius band [quoting Seifert-Threlfall, Lehrbuch der Topologie, Teubner, Leipzig, 1934] to distinguish between living and dead matter. *W. Feller* (Providence, R. I.).

Opatowski, I. On the motion of an electric particle. J. Math. Phys. Mass. Inst. Tech. 20, 418–424 (1941). [MF 6110]

The motion of an electric particle in magnetic and electric fields is investigated. By specializing the field to a certain extent and utilizing the energy and momentum integrals, the author obtains "forbidden" regions for the motion of the particle in the specialized but yet fairly general fields. This study is of particular interest in connection with cosmic rays and the motion of electrons in the earth's field. An example is worked out for the case of a magnetic field due to a magnetic dipole and a circular electric current, and of an electric field due to electrostatic charge uniformly distributed over a circle. *H. Poritsky.*

Ryder, Robert M. The electrical oscillations of a perfectly conducting prolate spheroid. J. Appl. Phys. 13, 327–343 (1942).

The qualitative features of the solutions of the equations

$$(1-\xi^2)X''(\xi)+(\alpha-\epsilon^2\xi^2)X(\xi)=0,$$
$$(\eta^2-1)Y''(\eta)+(\epsilon^2\eta^2-\alpha)Y(\eta)=0$$

are discussed. If $X_p(\xi)$ is the pth eigensolution of the first equation it gives the current distribution in the antenna for the pth mode of vibration and is an oscillatory function with $p+1$ nodes two of which are at the ends of the antenna while the others are in the interval $-(\alpha^{\frac{1}{2}}/\epsilon)<\xi<(\alpha^{\frac{1}{2}}/\epsilon)$.

At zero frequency ($\epsilon=0$) $X_p(\xi)(1-\xi^2)^{-1}$ is proportional to the associated Legendre function $P_{p,1}(\xi)$ and $\alpha_p=p(p+1)$. As ϵ increases from zero the parameter α also increases but so slowly that when $\epsilon=\frac{1}{2}p\pi$, $\alpha_p=\epsilon^2$. The function $X_p(\xi)$ is then sinusoidal. For $\epsilon>\frac{1}{2}p\pi$, $\alpha<\epsilon^2$ the curvature of the graph of $X_p(\xi)$ is away from the ξ-axis for $|\xi|>\alpha^{\frac{1}{2}}/\epsilon$ and the loops and nodes move in toward the center of the antenna. Qualitative sketches of $X_1(\xi)$ are given. For large values of η, $Y_p(\eta)$ approaches the value $Ae^{i\eta}+Be^{-i\eta}$, where A and B are constants. The notation Y is reserved for that solution which is asymptotically $e^{i\eta}$ as $\eta\to\infty$.

An integral equation and an integral form of the solutions give expressions for Y involving logarithms, or rather the function $Q_0(\eta)$. Series solutions of the equation for X are developed and the properties of the forced oscillations are considered. The properties of the pth mode and of the pth harmonic near resonance are next discussed and composite oscillations are considered. Two short tables are given in the appendix for $F(2p)$, Si $(2p\pi)$ and the radiative resistance R at resonance for $p=1(1)10$, where $F(2p)=\log(2p\pi)-\mathrm{Ci}\,(2p\pi)+C$, $R=(\mu/k)^{\frac{1}{2}}29.98F(2p)$. *H. Bateman.*

Buchholz, Herbert. Gekoppelte Strahlungsfelder im kreiszylindrischen Hohlleiter. Ann. Physik (5) **39**, 81–128 (1941). [MF 6233]

The paper deals with the radiation from a dipole (electric or magnetic) situated inside a perfectly conducting circular cylinder (wave-guide). The problem is to find a periodic Maxwellian field satisfying the following conditions: (i) the tangential component of the electric vector is to vanish on the surface of the cylinder; (ii) the field is to have an appropriate singularity at the dipole. In cylindrical coordinates, the field at (r, ϕ, z) due to a dipole at $(\rho, 0, 0)$ may be derived from the function e^{ikR}/R, where k is a constant proportional to the frequency and $R^2=z^2+r^2+\rho^2-2r\rho\cos\phi$. The boundary conditions are satisfied by superposition of the field due to the dipole and a suitable regular field, itself a sum of elementary fields involving Bessel functions. The essential step is, therefore, to express e^{ikR}/R suitably in terms of Bessel functions, and this is done, for $r>\rho$, by the formula

$$e^{ikR}/R=i(2a)^{-1}\sum_{n=0}^{\infty}\gamma_n\cos n\phi\int_{-\infty}^{\infty}J_n(s_\rho)H_n^{(1)}(s_r)\exp(isz/a)ds.$$

Here a is the radius of the cylinder and $s_r=(ir/a)(s^2-a^2k^2)^{\frac{1}{2}}$. Once the field due to a dipole has been found, the field due to a finite antenna is obtained by integration. If the cylinder is blocked by a perfectly conducting partition, images are taken with respect to its plane. Various particular cases are considered in detail, including the case of an antenna in the form of a closed loop.

The author does not refer to a paper on the same physical problem by Schelkunoff [Proc. I. R. E. **24**, 1388–1398 (1936)]. Schelkunoff used a Sommerfeld integral for e^{ikR}/R, and the formal developments of the two papers are different. *J. L. Synge* (Toronto, Ont.).

Chapman, S. and Ferraro, V. C. A. The geomagnetic ring-current: I. Its radial stability. Terr. Magnetism **46**, 1–6 (1941). [MF 6273]

The supposed ring-current, produced by ions flowing westward at a greater speed than the accompanying electrons, is replaced, for purposes of discussion, by a cylindrical current-sheet in a magnetic field parallel to the axis of the cylinder and varying inversely as the cube of the distance from this axis. The condition for stability is found to be $H>-(3/2)H'$, where H is the value of the permanent magnetic field at the position of the sheet and H' is the intensity of the field produced by the sheet. The displacements are all supposed to be radial and the period of radial oscillations is found. In one numerical illustration it is one minute; in another it is ten minutes. The associated magnetic variations are also estimated. A periodic magnetic variation of range 20γ when $H=-40\gamma$ is thought to correspond to a variation in the radius of the sheet by ±1 mile. *H. Bateman* (Pasadena, Calif.).

Chang, M. T. Quasi-holonomic transformation and non-holonomic reference system in electrical rotating machinery. Sci. Rep. Nat. Tsing Hua Univ. (A) **4**, 173–184 (1940). [MF 5325]

The author discusses the tensor transformation of Lagrange's equations to quasi-coordinates, and also the imposition of non-holonomic constraints. A number of misprints make the argument obscure in places, but the formalism appears to follow the usual lines. A transformation matrix for an n-phase alternator is given. *J. L. Synge* (Toronto, Ont.).

Müller, J. J. et Rostas, E. Un générateur à temps de transit, utilisant un seul resonateur de volume. Helvetica Phys. Acta **13**, 435–450 (1940).

An analysis is made of the performance of a generator consisting of a long vacuum tube surrounded at one end by a cavity resonator. Optimum values are given for the parameters of such a system. Several advantages are claimed for this type as compared with the well-known Klystron oscillators. *R. M. Foster* (New York, N. Y.).

Lüdi, F. Zur Theorie der Laufzeitschwingungen. Helvetica Phys. Acta **13**, 77–121 (1940).

The transit time oscillations which are of particular importance in the study of ultra-short waves are regarded as a coupled system of oscillating space charge and electric oscillating circuit. The differential equations governing such a system are similar to those for a two circuit transmitter. For a self regulating system, the stationary solutions of the differential equations help explain space charge effects such as coupling waves, intensity effects, etc. *A. E. Heins.*

Reed, Myril B. Matrices, tensors, or dyadics for studying electrical networks? J. Appl. Phys. **12**, 773–779 (1941).

The author discusses the relative merits of dyadic algebra, tensor analysis and matrix algebra, as applied in electrical network theory, particularly in the study of the transformation of a network into equivalent structures. He concludes that, of the three possibilities, matrix methods are most easily and naturally adapted to the treatment of this problem. *R. M. Foster* (New York, N. Y.).

Ghizzetti, Aldo. La trasformazione di Laplace e il calcolo simbolico degli elettrotecnici. Univ. e Politecnico Torino. Rend. Sem. Mat. Fis. **7**, 83–99 (1941). [MF 6280]

This is a short expository representation of the symbolic calculus of electric circuit theory based on the Laplace transformation and following the customary pattern. *H. G. Baerwald* (Cleveland, Ohio).

Hartley, R. V. L. A more symmetrical Fourier analysis applied to transmission problems. Proc. I. R. E. **30**, 144–150 (1942). [MF 6607]

The author uses a different transform relationship from the usual Fourier one. Defining

$$\operatorname{cas} x = \cos x + \sin x,$$

his relationship is

$$f(t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \psi(w) \operatorname{cas} wt\, dw,$$

$$\psi(w) = (2\pi)^{-1} \int_{-\infty}^{\infty} f(t) \operatorname{cas} wt\, dt.$$

He interprets this relationship physically and recasts several relationships and methods in circuit theory in terms of his transforms.

While in certain situations the author's transform relationship may be more convenient than the usual Fourier relationships, the reviewer believes that in general the usual Fourier transforms will continue to dominate linear circuit theory. [The author refers in footnote 10 to Bode in 1940 as having discovered the relationship between the amplitude and the phase of the steady state characteristics of a linear system. Actually this relationship is that between conjugate functions in complex variables and is connected by the so-called Hilbert transform relationship as Wiener pointed out and as is very explicitly related by Y. W. Lee [J. Math. Phys. Mass. Inst. Tech. **11**, 83–113 (1932)] by means of the Fourier transforms of Laguerre functions.]

N. Levinson (Cambridge, Mass.).

Nakasima, Akira and Hanzawa, Masao. Expansion theorem and design of two terminal relay networks. I. Nippon Elec. Commun. Engrg., no. **24**, 203–210 (1941). [MF 4517]

This is an application of the two-valued Boolean algebra to the theory of relay and switching circuits. The impedance of a closed circuit is designated by the element 0, and an open circuit by 1. Two electrical elements in series are represented by the Boolean sum, and in parallel by the product. If a circuit x is open, there is a circuit x' which is closed, and vice versa. Relays provide a functional relationship among the various elements. The electrical analogue of the expansion theorem

$$f(x_1, x_2, \cdots, x_n) = x_1 f(1, x_2, \cdots, x_n) + x_1' f(0, x_2, \cdots, x_n)$$

is made the basis for a study of the synthesis of relay and switching circuits, making extensive use of symmetric functions. A similar application of Boolean algebra has been made by C. E. Shannon [Trans. Amer. Inst. Elec. Engrs. **57**, 713–723 (1938)]. *R. M. Foster* (New York, N. Y.).

Pipes, Louis A. The transient behaviour of four-terminal networks. Philos. Mag. (7) **33**, 174–214 (1942). [MF 6447]

This is an exposition of the application of the "operational calculus," as based on the Fourier-Mellin theorem, to the steady state and transient treatment of passive two-terminal-pair networks. The treatment is conventional throughout and covers the derivation of the quadripole equations from the network matrix, application of the Laplace transformation, simple special circuits, terminations, lumped and distributed recurrent structures. Conditions for the validity of the basic theorem and a collection of operational rules and transforms are given in the appendix.

H. G. Baerwald (Cleveland, Ohio).

Pipes, Louis A. Linear transformations in three-phase circuits. Trans. Amer. Inst. Elec. Engrs. **60**, 351–356 (1941).

It is pointed out that the method of symmetrical components and its modifications are special cases of linear transformations of the voltages and currents of three-phase systems. These special cases attain the practical aim, that is, (full or partial) diagonalization of the impedance matrix of the system and thus resolution into eigenphenomena only if the three-phase system itself shows certain symmetries: in case of symmetrical components, it must be invariant under cyclic permutation of the phases; in the Clarke and Kimbark systems, it must be symmetrical about one phase. For general systems, transformation to diagonal form involves the numerical ratios of the impedance components and can be carried out in practice by stepwise application of elementary transformations. A numerical example is given. Invariance of watt-power under linear transformations requires the current transform to be equal to the conjugate transposed voltage transform, which relation holds for the symmetrical components, but not for the Clarke and Kimbark transforms. *H. G. Baerwald.*

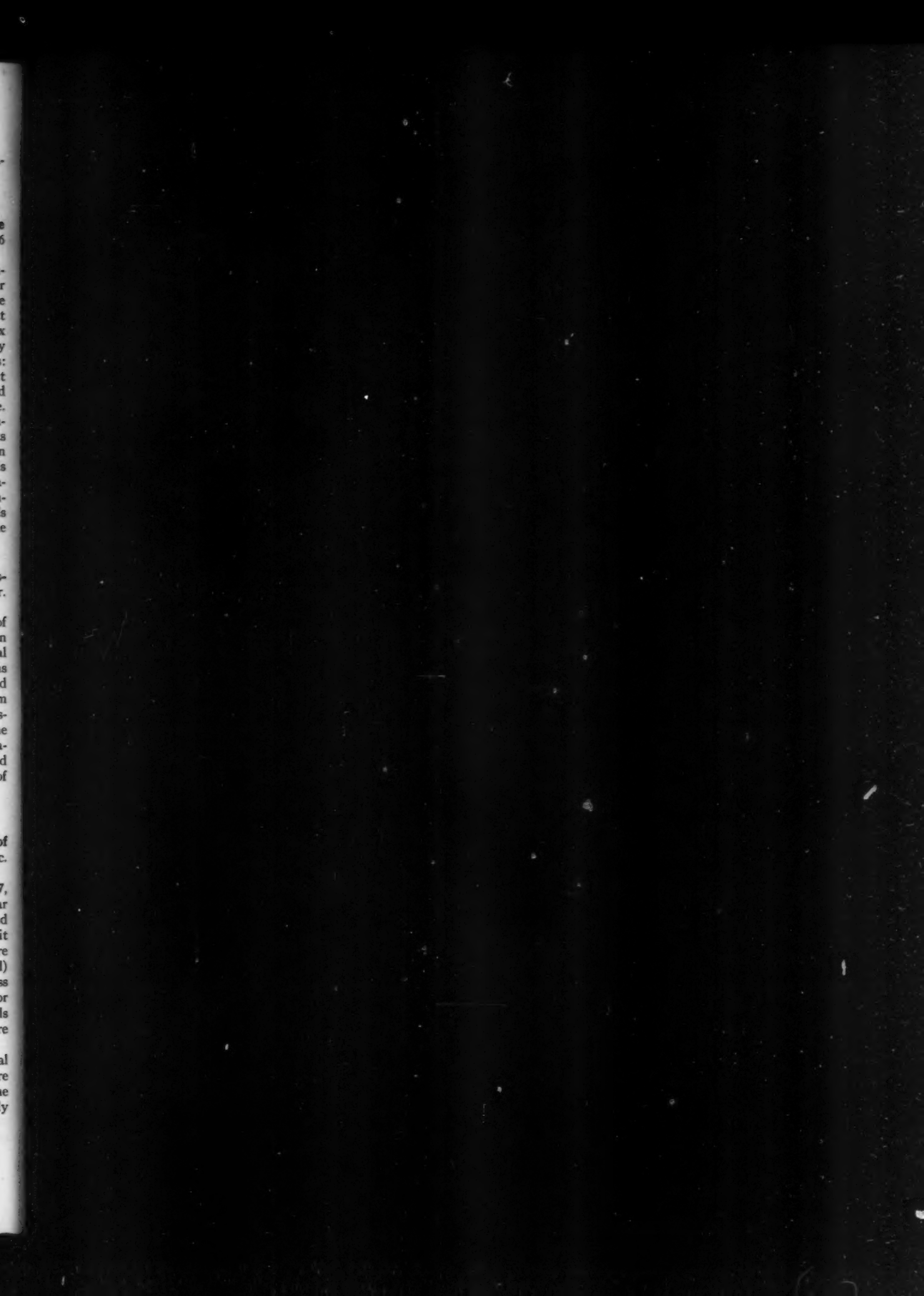
Pipes, Louis A. Transient analysis of completely transposed multiconductor transmission lines. Trans. Amer. Inst. Elec. Engrs. **60**, 346–350 (1941).

The distributed impedance and admittance matrices of an equally transposed bundle of n uniform transmission lines are totally symmetrical, that is, have equal diagonal and equal nondiagonal elements. Therefore, the system has only two eigenphenomena, one of which is $(n-1)$-fold degenerate, and the voltages and currents of the system are represented in terms of them by means of a cyclic transformation. This represents a certain generalization of the method of symmetrical components. A Laplace transformation is applied to the equations of the eigenphenomena, and thus closed expressions are obtained for the propagation of nonstationary phenomena along the bundle.

H. G. Baerwald (Cleveland, Ohio).

Higgins, Thomas James. Formulas for the inductance of rectangular tubular conductors. Trans. Amer. Inst. Elec. Engrs. **60**, 1046–1050 (1941).

Dwight and Wang [Trans. Amer. Inst. Elec. Engrs. **57**, 762–765 (1938)] gave formulas for thin square tubular conductors. The present paper removes the "thin" and "square" restrictions by computing the inductance per unit length of tubular rectangular conductors which are square cornered (corners of commercial conductors are rounded) and carry currents uniformly distributed over their cross sections. End effects are neglected. Results are given for two conductors and may be extended by known methods to cover larger numbers provided that the conductors are axially coplanar.

The method of calculation is by way of vector potential to total energy to inductance. Double Fourier integrals are used in the solution of the Poisson-type equation for the vector potential. Results are finally reduced to practically convenient, rapidly convergent power series.

J. L. Barnes (Medford, Mass.).